Recitation Guide - Week 2

Topics Covered: Contrapositive, Contradiction, Truth Table, Combinatorial Proof

Problem 1: Let \( a, b \in \mathbb{Z} \) and \( n \) is a positive integer. If \( n \) does not divide \( ab \), then \( n \) does not divide \( a \) and \( n \) does not divide \( b \).

Solution:

The contrapositive: If \( n \) divides \( a \) or \( n \) divides \( b \), then \( n \) divides \( ab \). To prove this, assume \( n|a \) without loss of generality. Then \( a = kn \) for some \( k \in \mathbb{Z} \). Hence \( ab = knb = (kb)n \). Since \( k, b \in \mathbb{Z} \), we know that \( kb \) must be an integer, so \( n|ab \).

Problem 2: Prove that \( \sqrt{6} \) is irrational.

Solution:

We will use a proof by contradiction. Assume for the sake of contradiction that \( \sqrt{6} \) is rational. This means that we can express \( \sqrt{6} \) as the following: \( \frac{a}{b} \) where \( a \) and \( b \) are relatively prime natural numbers and \( b \neq 0 \). This means that \( 6 = \frac{a^2}{b^2} \Rightarrow 6b^2 = a^2 \).

If \( 6|a^2 \), then \( 2|a^2 \) which implies that \( a \) must be even (recall the Lemma proved in Lecture 4). Because \( a \) is even, let \( a = 2c \) where \( c \) is some integer.

\[
\begin{align*}
6b^2 &= a^2 \\
(2)(3)b^2 &= (2c)^2 \\
(2)(3)b^2 &= (2)(2)c^2 \\
3b^2 &= 2c^2
\end{align*}
\]

If \( 2|(3b^2) \), then \( 2|b^2 \) which implies that \( b \) must be even (see Lemma above). So, clearly, \( a \) and \( b \) are both even. However, this presents a contradiction: \( a \) and \( b \) must be relatively prime natural numbers, and thus cannot both be even (divisible by 2).

Problem 3:

Which of the following are logically equivalent: \( p \land \neg(\neg p \land \neg q) \), \( (\neg p \land \neg q) \Rightarrow q \), \( p \lor C \)

Solution:

Consider the following truth tables:
\begin{align*}
|S| &= \sum_{k=0}^{r} \binom{n}{k} \binom{m}{r-k} \\
&= \sum_{k=0}^{r} \binom{n}{k} \binom{m}{r-k}
\end{align*}

which gives us the left hand side of the expression.