Topics Covered: $r$-Combinations with Repetition Allowed (Stars & Bars), Combinatorial Proofs, Binomial Theorem

Problem 1: Prove that $4^n - 1$ is divisible by 3 for all $n \in \mathbb{Z}^+$. 

Solution:

We proceed by induction on $n$.

**Induction Hypothesis:** Assume the claim is true when $n = k$, for some $k \geq 1$, $k \in \mathbb{Z}^+$. In other words, we assume

$$4^k - 1 = 3m$$

For some integer $m$.

**Base Case:** $n = 1$ In this case, we have $4^1 - 1 = 3$, which is divisible by 3.

**Induction Step:** We want to prove the claim is true for $n = k + 1$. That is, we want to show

$$4^{k+1} - 1 = 3\ell$$

For some integer $\ell$. We can do this as follows:

$$4^{k+1} - 1 = 4(4^k) - 1$$

$$= 4(4^k - 1) + 4 - 1$$

$$= 4(3m) + 3$$

(By I.H)

$$= 12m + 3$$

$$= 3(4m + 1)$$

$$= 3\ell$$

Since $m$ is an integer by the induction hypothesis, $\ell = 4m + 1$ is an integer, thus showing the claim is true for $n = k + 1$.
Problem 2: Bharath has selected 100 schools to which he is going to donate a total of 1000 indistinguishable blue-and-gold (Warriors colors) scarves for the children.

(a) How many ways can he distribute his scarves?
(b) What if he insists that each school get at least 5 scarves?
Solution:

(a) This is a standard stars and bars/sticks and crosses problem. There are 100 categories (the schools) to which we distribute 1000 indistinguishable objects (the scarves). Thus we consider the number of arrangements of 1000 stars with 100 − 1 = 99 bars, which is

\[ \binom{1000 + 100 - 1}{1000} = \binom{1099}{99} \]

(b) If each school gets at least 5 scarves, we begin by distributing exactly 5 scarves to each school before we distribute the rest to satisfy the condition. We are left with 1000 − 5 \times 100 = 500 scarves to distribute. We count the number of arrangements of scarves now by simply applying stars and bars once more with 500 stars and 100 bars. Thus the final answer is

\[ \binom{500 + 100 - 1}{500} = \binom{599}{99} \]
Problem 3: The “elevator operator” can distinguish between cats and dogs, but all cats look the same to him, and all dogs look the same to him. 10 cats and 10 dogs get on the elevator on the ground floor (Floor 0). By the time the elevator gets to Floor 6, the elevator is empty of all animals. How many ways could the cats and dogs get off the elevator? Note that cats are indistinguishable from one another and dogs are indistinguishable from one another.
Solution:

We can think of this as a multiset problem with two parts. We can break it down into separate stars and bars problems and combine them at the end.

There are 6 floors in which we distribute 10 cats. Since the cats are indistinguishable, they will represent the stars in this problem. We now also have $6 - 1 = 5$ bars to represent the floors where they get off. The number of arrangement of cats is then

$$\binom{10 + 6 - 1}{10} = \binom{15}{10}$$

Similarly, we can do the same for the dogs. 10 indistinguishable objects distributed along 6 slots will give us

$$\binom{10 + 6 - 1}{10} = \binom{15}{10}$$

But this isn’t the end! We still have to combine them. We can do this by using the multiplication rule, since the order in which the cats exit the elevator is independent of the order in which dogs exit the elevator.

Step 1: Choose the order in which cats exit the elevator. From above, there are $\binom{15}{10}$ ways.

Step 2: Choose the order in which dogs exit the elevator. Again, there are $\binom{15}{10}$ ways.

which gives us

$$\binom{15}{10} \times \binom{15}{10} = \binom{15}{10}^2$$
Problem 4:

Give a combinatorial proof for the following, where $m \leq n$:

\[
\sum_{k=0}^{m} \binom{n}{k} \binom{n-k}{m-k} = 2^m \binom{n}{m}
\]
Solution:

Consider the following counting problem:

Given a set of \( n \) people, how many ways are there to hire \( m \) people such that any number of those \( m \) people can also be designated as managers?

(RHS) We use two steps to hire people and designate managers.

Step 1: Choose the \( m \) people who are hired out of the \( n \) job applicants.
Step 2: Designate any subset of the \( m \) people as managers.

In Step 1, we are simply choosing \( m \) out of \( n \) items. Thus there are \( \binom{n}{m} \) ways to do Step 1. In Step 2, we are taking a subset of \( m \) items, so there are \( 2^m \) ways to do Step 2. Applying the multiplication rule, there are \( 2^m \binom{n}{m} \) ways to hire people and designate managers.

(LHS) Let \( S \) be a set that includes all of the ways that we can hire people and designate managers. We can partition \( S \) into sets \( S_0, S_1, S_2, \ldots, S_m \) where set \( S_k \) (\( 0 \leq k \leq m \)) represents all of the ways that we can hire \( m \) people and designate exactly \( k \) managers. For each \( k \), \( |S_k| \) can be calculated in the following way:

Step 1: Choose the \( k \) managers that we want from the \( n \) total people.
Step 2: Hire people from the remaining \( n - k \) people so that we end up with a total of \( m \) employees.

In Step 1, we are simply choosing \( k \) people out of \( n \), so there are \( \binom{n}{k} \) ways to do Step 1. In Step 2, we must hire additional people so that we have a total of \( m \) employees. Since we have already hired \( k \) people, we can only hire \( m - k \) more people. In addition, we cannot choose any of those \( k \) people to hire (since they have already been hired), so there are \( n - k \) people to choose from. Thus, there are \( \binom{n-k}{m-k} \) ways to do Step 2.

Applying the multiplication rule, there are \( \binom{n}{k} \binom{n-k}{m-k} \) ways to hire \( m \) people and designate exactly \( k \) managers.

Thus, the total number of ways to hire \( m \) people and designate any number of managers is

\[
|S| = \sum_{k=0}^{m} |S_k| = \sum_{k=0}^{m} \binom{n}{k} \binom{n-k}{m-k}
\]

which is the LHS.