Topics Covered: Probability

Problem 1:
Compute the probability of the event “when we roll $n$ (distinguishable) fair dice any $k$ of the dice show the same number while the other $n-k$ show numbers different from the one shown by the $k$ dice”. Assume $n \geq 3$ and $\frac{n}{2} < k < n$.

Solution:
As discussed in class, we have a uniform probability space whose outcomes are sequences of length $n$ of numbers from $[1..6]$. In other words, the sample space is given by the cartesian product of $[1..6] \times \cdots \times [1..6]$ ($n$ times), i.e., $\Omega = [1..6]^n$. By the Product Rule, there are $6 \times \cdots \times 6 = 6^n$ such sequences so each outcome has probability $\frac{1}{6^n}$.

Let $E$ be the event where exactly $k$ of the dice show the same number. We see that we are trying to find $\Pr(E)$. To compute the desired probability it suffices, by Proposition 13.6, to count the cardinality of $E$, i.e., the number of sequences (of interest) in which $k$ positions have the same number from $t \in [1..6]$ while the other $n-k$ position show numbers different from $t$. Such a sequence can be constructed as follows:

- Step 1: Choose $t \in [1..6]$. This can be done in 6 ways.
- Step 2: Choose $k$ of the $n$ positions in the sequence. This can be done in $\binom{n}{k}$ ways.
- Step 3: Place $t$ in each of these positions. This can be done in 1 way.
- Step 4: For each of the remaining $n-k$ positions choose a number from $[1..6] \setminus \{t\}$. We see that there are 5 such numbers, and $n-k$ positions that we have left to fill. Thus, this can be done in $5^{n-k}$ ways.

By the Product Rule, the number of sequences of interest is $6 \binom{n}{k} 5^{n-k}$. Hence, the probability we are asked for is given by:

$$\Pr(E) = \frac{|E|}{|\Omega|} = \left(6 \binom{n}{k} 5^{n-k}\right) \frac{1}{6^n} = \binom{n}{k} \frac{5^{n-k}}{6^{n-1}}$$
**Problem 2**  Compute the probability of the event “when we roll two identical beige dice the numbers add up to an even number”.

**Solution:**

Recall Example 13.3 and the discussion following Definition 13.4.

We first observe, by our discussion in lecture, the sample space for this problem is given by:

$$
\Omega = \{ \{x, y\} \mid x, y \in [1..6]\} \cup \{x-x \mid x \in [1..6]\}
$$

Let $E$ be the event where the sum of the two rolls results in an even number. Note that we have $\binom{6}{2} = 15$ outcomes in which the dice show different numbers; each of these has probability $\frac{1}{18}$ by our analysis in lecture. Among these outcomes, the numbers add up to an even number if they are both odd, and there are 3 of these, $\{1, 3\}, \{1, 5\}, \{3, 5\}$, or if they are both even – there also 3 of these: $\{2, 4\}, \{2, 6\}, \{4, 6\}$. So that’s 6 outcomes of probability $\frac{1}{18}$ each in which the numbers are different.

We also have 6 more outcomes in which the die show the same number; each of these has probability $\frac{1}{36}$, again from lecture. In all these outcomes the numbers add up to an even number, hence we have another 6 outcomes of probability $\frac{1}{36}$ each.

We now calculate the desired probability using the definition of event (Definition 13.5):

$$
\Pr(E) = \sum_{w \in E} \Pr(w)
= \Pr(\{1, 3\}) + \Pr(\{1, 5\}) + \Pr(\{3, 5\}) + \Pr(\{2, 4\}) + \Pr(\{2, 6\}) + \Pr(\{4, 6\}) + \sum_{x \in [1..6]} \Pr(x-x)
= 6 \times \frac{1}{18} + 6 \times \frac{1}{36}
= \frac{1}{3} + \frac{1}{6}
= \frac{1}{2}
$$

That’s the same answer as in Example 13.7! And indeed, as explained in Example 13.7, since “adding up to even” is an event in which the die color doesn’t matter, we could have provided a solution that assumes the die are green-purple rather than beige-beige.
Problem 3  Let \( A, B, C \) be three events in the same probability space. Show that

\[
\Pr(A \cup B \cup C) = \Pr(A) + \Pr(B) + \Pr(C) - \Pr(A \cap B) - \Pr(B \cap C) - \Pr(C \cap A) + \Pr(A \cap B \cap C)
\]

\( \text{(Hint: apply (P6) thrice.)} \)

As a reminder, \((P6)\) \(\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)\)

Solution:

By a first application of (P6) we obtain:

\[
\Pr(A \cup B \cup C) = \Pr(A \cup (B \cup C)) = \Pr(A) + \Pr(B \cup C) - \Pr(A \cap (B \cup C))
\]

Combine this with a second application of (P6)

\[
\Pr(B \cup C) = \Pr(B) + \Pr(C) - \Pr(B \cap C)
\]

Now, convince yourselves by diagram that \( A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \). Also that

\[
(A \cap B) \cap (A \cap C) = A \cap B \cap C
\]

A third application of (P6) gives us

\[
\Pr((A \cap B) \cup (A \cap C)) = \Pr(A \cap B) + \Pr(A \cap C) - \Pr((A \cap B) \cap (A \cap C))
\]

\[
= \Pr(A \cap B) + \Pr(A \cap C) - \Pr(A \cap B \cap C)
\]

Putting together the three equations we obtain the desired identity.
Problem 4
We have three wooden buckets, $T_A, T_B, T_C$ and we throw $n \geq 3$ metal keys in them. The key throws are mutually independent and each key is equally likely to land in each of the three buckets.

(a) Let $A$ be the event that after all keys are thrown bucket $T_A$ has at least one key in it and similarly associate an event $B$ with $T_B$. Are $A$ and $B$ independent? Justify your answer.

(b) Compute the probability that after all keys are thrown, each of the three buckets has at least one key in it. Justify your answer.

Solution:

(a) For $i = 1, \ldots, n$ let $A_i$ be the event that key $i$ is thrown in bucket $T_A$. We have $\Pr(A_i) = \frac{1}{3}$.

Clearly $A = A_1 \cup \cdots A_n$ and since the events $A_1, \ldots, A_n$ are mutually independent we can compute (see Proposition 14.10):

$$\Pr(A) = \Pr(A_1 \cup \cdots A_n) = 1 - \prod_{i=1}^{n} (1 - \Pr(A_i)) = 1 - \left(1 - \frac{1}{3}\right)^n = 1 - \left(\frac{2}{3}\right)^n$$

Similarly, $\Pr(B) = 1 - \left(\frac{2}{3}\right)^n$. To check independence we also need $\Pr(A \cap B)$.

Upon reflection, we notice that there is one aspect of the problem that we have not used yet: the keys get thrown only in $T_A, T_B$ and $T_C$. Thus, $A \cap B$, which means that both $T_A$ and $T_B$ are empty after all keys are thrown, is the same as the event “all keys get thrown in $T_C$” and therefore, by mutual independence, has probability $\left(\frac{1}{3}\right)^n$, as each key has a $\frac{1}{3}$ probability of being thrown into $T_C$. Now we can compute, using properties of probability and De Morgan’s Laws:

$$\Pr(A \cap B) = \Pr(A) + \Pr(B) - \Pr(A \cup B)$$
$$= \Pr(A) + \Pr(B) - (1 - \Pr(A \cap B))$$
$$= 1 - \left(\frac{2}{3}\right)^n + 1 - \left(\frac{2}{3}\right)^n - \left(1 - \left(\frac{1}{3}\right)^n\right)$$
$$= 1 - 2\left(\frac{2}{3}\right)^n + \left(\frac{1}{3}\right)^n$$

But we also know that:

$$\Pr(A) \cdot \Pr(B) = \left(1 - \left(\frac{2}{3}\right)^n\right) \left(1 - \left(\frac{2}{3}\right)^n\right) = 1 - 2\left(\frac{2}{3}\right)^n + \left(\frac{4}{9}\right)^n$$

Since $\frac{1}{3} \neq \frac{4}{9}$ it follows that $\Pr(A) \cdot \Pr(B) \neq \Pr(A \cap B)$ hence $A$ and $B$ are not independent.

(b) We continue with the notation introduced in part (a) and we also define $C$ to be the event “$T_C$ is not empty after all keys are thrown.” This part asks for $\Pr(A \cap B \cap C)$. We are tempted to multiply probabilities but we do not know if $A, B, C$ are mutually independent. In fact, in part (a) we saw that $A \not\perp B$. Although it is still possible that $\Pr(A \cap B \cap C) = \Pr(A) \cdot \Pr(B) \cdot \Pr(C)$ (see Example 14.6) there is no reason to hope for this here (and in fact we shall see that it does not hold).
Instead, we will use the Principle of Inclusion-Exclusion for three events (see Proposition 13.18 and Problem 3 above!):

\[
\text{Pr}(A \cup B \cup C) = \text{Pr}(A) + \text{Pr}(B) + \text{Pr}(C) - \text{Pr}(A \cap B) - \text{Pr}(B \cap C) - \text{Pr}(C \cap A) + \text{Pr}(A \cap B \cap C)
\]

Since we have at least one key, at least one of the buckets ends up non-empty. Hence \(A \cup B \cup C = \Omega\), meaning \(A \cup B \cup C\) consists of all the outcomes and has probability 1. From part (a) we have:

\[
\text{Pr}(A) = \text{Pr}(B) = \text{Pr}(C) = 1 - \left(\frac{2}{3}\right)^n
\]

\[
\text{Pr}(A \cap B) = \text{Pr}(B \cap C) = \text{Pr}(C \cap A) = 1 - 2 \left(\frac{2}{3}\right)^n + \left(\frac{1}{3}\right)^n
\]

We plug in and obtain

\[
\text{Pr}(A \cap B \cap C) = \text{Pr}(A \cup B \cup C) - \text{Pr}(A) - \text{Pr}(B) - \text{Pr}(C) + \text{Pr}(A \cap B) + \text{Pr}(B \cap C) + \text{Pr}(C \cap A)
\]

\[
= 1 - 3 \left(1 - \left(\frac{2}{3}\right)^n\right) + 3 \left(1 - 2 \left(\frac{2}{3}\right)^n + \left(\frac{1}{3}\right)^n\right)
\]

\[
= 1 - 3 \left(\frac{2}{3}\right)^n + 3 \left(\frac{1}{3}\right)^n
\]

BTW, for \(n = 3\) this gives \(\frac{2}{9}\).