Recitation Guide - Week 2

**Topics Covered:** Sets, Counting, Proofs

**Multiplication Rule:**

If a procedure can be broken down into $k$ steps, where the $i^{th}$ step can be done in $n_i$ ways (where each $n_i$ is independent of the preceding steps), the entire procedure can be done in $n_1 \times n_2 \times \cdots \times n_k$ ways.

**Problem 1:**

1. List the members of these sets:
   (a) $\{x \mid 2x \text{ is a positive integer less than 9}\}$
   (b) $\{x \mid x \text{ is the cube of a natural number, and } x < 100\}$
   (c) $\{x \mid x \text{ is a positive integer such that } x^2 = 9 \text{ and } x < 3\}$

2. What is the cardinality of each of the following sets?
   (a) $\{a\}$
   (b) $\{a, \{a\}, \emptyset\}$
   (c) $\{a, \{a\}, \{a, b\}\}$
   (d) $2^{\{a\}}$

3. Determine whether each of the following is true or false:
   (a) $\{x\} \subseteq \{x\}$
   (b) $\{x\} \subset \{x\}$
   (c) $x \in \{\{x\}\}$

**Solution:**

1. (a) $\left\{\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}, 4\right\}$
   (b) $\{0, 1, 8, 27, 64\}$
   (c) $\emptyset$

2. (a) 1
   (b) 3
   (c) 3
   (d) 2. The powerset of $\{a\}$ is $\{\{a\}, \emptyset\}$ since there is only one element.

3. (a) True
   (b) False
   (c) False
Problem 2: Assume that $A \subseteq B \subseteq C$. Prove that $(C \setminus B) \cup (B \setminus A) \subseteq C \setminus A$. (You are NOT allowed to use in the proof set algebra facts such as $A \subseteq B \iff A \cup B = B$ or $A \setminus A = \emptyset$. Your proof should use only the definitions of subset, set difference, union, and empty set and logical manipulation of statements.)

Solution:

Let $x \in (C \setminus B) \cup (B \setminus A)$. W.T.S. (want to show) $x \in C \setminus A$. Since $x \in (C \setminus B) \cup (B \setminus A)$, we know that $x \in C \setminus B$ or $x \in B \setminus A$. We thus consider the following two cases:

Case 1: $x \in C \setminus B$.

Since $x \in C \setminus B$, we see by definition of set difference that $x \in C$ and $x \notin B$. However, in order to show that $x \in C \setminus A$, we must also show that $x \notin A$.

Note that, since $A \subseteq B$, we know $\forall a, a \in A \implies a \in B$. Since this claim is true, we know that its contrapositive is also true, namely that $\forall a, a \notin B \implies a \notin A$. Intuitively, if some element $a$ is not an element of $B$, then $a$ also cannot be an element of $A$, as every element of $A$ must be in $B$.

Combining this with the fact that $x \notin B$, we see that $x \notin A$. Thus, we conclude that $x \in C \setminus A$, as desired.

Case 2: $x \in B \setminus A$.

Since $x \in B \setminus A$, we see that $x \in B$ and $x \notin A$, as above. Furthermore, because $B \subseteq C$, we immediately have that $x \in C$. Therefore, $x \in C \setminus A$.

In both cases $x \in C \setminus A$. We have therefore shown that $(C \setminus B) \cup (B \setminus A) \subseteq (C \setminus A)$.
Problem 3:

Your favorite pizza place in the world, Nikhil's Thicc Nicc's Pizzeria, is known for its variety of different pizzas. Thicc Nicc's has 5 different kinds of tomato sauces and 6 different kinds of cheese. In addition, you can add one of any 20 different toppings, which are optional. On top of all of these choices, you can choose a thin crust or a thicc crust. How many different pizzas can you possibly order from Thicc Nicc's?

Solution:

We will apply the Multiplication Rule:

   Step 1: Choose the sauce. (5 ways)
   Step 2: Choose the cheese. (6 ways)
   Step 3: Choose the topping. (20 toppings + 1 option for no toppings = 21 ways)
   Step 4: Choose the crust. (2 ways)

Multiplying all of these, we get $5 \times 6 \times 21 \times 2 = 1260$ pizzas.
Problem 4:
In how many ways can we seat 5 girls and 4 boys around a circular table such that no two boys are neighbors?

Solution:
Let’s start by seating any girl at the table. Once we have her seated, we can label her seat 1, the seat to her left 2, the one to the left of that 3, and so on, until we have the chair immediately to her right labeled chair 9. We now fill in the remaining odd number chairs with girls and the remaining even number chairs with boys. Notice that this way between every pair of girls we fill in the middle chair with a boy, ensuring that no two boys are neighbors. We can now count using the Multiplication Rule.

Step 1: Choose the first girl. (5 ways)
Step 2: Fill the odd seats with the remaining girls. (4! ways)
Step 3: Fill the even seats with boys. (4! ways)

Multiplying all of these, we get $5 \times 4! \times 4! = 2880$ ways.

Figure 1: Assign the girls to the odd seats, starting with $G_1$, and working your way around. Then fill in the even seats with the boys.

Alternate Solution:
First, we will find the number of ways to arrange 5 girls in a circle. Now, there will be 5 “gaps” between those 5 girls. Each boy must be in one of these gaps, and no two can be in the same gap (or else they would be sitting next to each other!). We’ll then choose a gap to be empty and seat the boys in the remaining spots. By Multiplication Rule:

Step 1: Arrange the 5 girls in a circle. \(\frac{5!}{5\text{ ways}}\)
Step 2: Choose one of the 5 “gaps” to be empty. (5 ways)
Step 3: Arrange the 4 boys in the other gaps. (4! ways)
Multiplying all of these, we get $4! \times 5 \times 4! = 2880$ ways.

Here’s why there are $\frac{5!}{5}$ ways to do Step 1: Let’s first arrange 5 elements in a line. There are 5! ways of doing this. If we connect the last element in the line to the first element, we make a circle with 5 elements. So that should be 5! ways to arrange the 5 girls in a circle, right? Not quite! While this approach works for a line, a circle is continuous so it wouldn’t be 5! for a circle. Consider the following arrangements: 1-2-3-4-5, 2-3-4-5-1, 3-4-5-1-2, 4-5-1-2-3, and 5-1-2-3-4. These are all the same arrangement in a circle, even though they start at different numbers! So we’re overcounting here by a factor of 5, because for every permutation, there are 5 rotations that give us the same circle. To get our final answer, we will therefore divide the total number of permutations, 5!, by 5, to get 4!. So there are 4! ways to do Step 1. Phew!

Note that this arrangement is a little “special,” since there is only one spot where two girls are neighbors. You might consider whether we can do the same thing when there are 5 girls and 5 boys. Try it as an exercise!