Recitation Guide - Week 11

Topics Covered: Graphs

Problem 1:
Let $T$ be a tree where the maximum degree is $\Delta$. Prove that $T$ has at least $\Delta$ leaves.

Solution:
We will use the (non-standard) notation $\lambda(T)$ to denote the number of leaves in a tree $T$. Thus, we can rewrite the claim as $\lambda(T) \geq \Delta(T)$.

Direct Proof:
Let $v \in V$ have degree $\Delta$ in $T = (V, E)$. Consider the subgraph induced on the vertices $V \setminus \{v\}$. Each neighbor of $v$ is in a distinct component in this graph, because we have destroyed the unique path between any two of $v$’s neighbors in $T$. Thus there are $\Delta$ components, each of which is a tree.

There are two possibilities for each component. If a component is a single node, then this single node is a leaf adjacent to $v$ in $T$. If the component has at least 2 nodes, then it has at least 2 leaves. One of the leaves may be adjacent to $v$ and not a leaf in $T$. But the other leaf in this component is still a leaf in $T$. In any case, each component contains at least one leaf of $T$ and hence $T$ must have $\Delta$ leaves.

Maximal Path:
Let $v \in V$ have degree $\Delta$. For each $u_i, u_j \in N(v)$, let $P_{i,j}$ be a maximal path including $u_i - v - u_j$. Note that there must be at least $\left(\frac{\Delta}{2}\right)$ such paths, since any pair of starting edges gives a different path. We know that any such path $P_{i,j}$ must terminate in two leaves (call them $w_{i,j}$ and $x_{i,j}$). Lastly, note that since there is a unique path between any two vertices in a tree, every pair of leaves admits at most one maximal path. If there were $\lambda(T) < \Delta$ leaves, we would only have $\left(\frac{\lambda(T)}{2}\right) < \left(\frac{\Delta}{2}\right)$ distinct maximal paths, a contradiction; we must then have $\lambda(T) \geq \Delta$.

Contradiction:
Assume that $\Delta \geq 2$, since the cases of $\Delta = 0$ and $\Delta = 1$ are clearly true. Suppose for the sake of contradiction that there are at most $\lambda(T) < \Delta$ leaves. For each $u_i \in N(v)$, let $p_i$ be a maximal path beginning with $v, \{v, u_i\}, u_i$. Note that there must be $\Delta$ such paths. We know from Definition 19.13 that any such path $p_i$ must terminate in a leaf $\ell_i$.

By the Pigeonhole Principle, where the pigeons are the terminating leaves of each path and the holes are the $\lambda(T)$ leaves available, we know that, since $\lambda(T) < \Delta$, two paths share the same terminating leaf, say $\ell_\omega$.

This is a contradiction, since the path between $\ell_\omega$ and $v$ are unique in a tree.

Induction on the number of vertices:
Let us prove this by induction on the number of vertices in the graph $n$.

We formulate a proposition $P(n)$ which is: in a tree with $n$ vertices and maximum degree $\Delta$, the number of leaves in the tree is at least $\Delta$. 

Base Case \((n=0, 1, 2 \text{ and } 3)\): The claim holds vacuously for \(n = 0\). The case of \(n = 1\) is trivial - a graph of just 1 node has maximum degree 0 and 0 leaves. There is only one possible tree when \(n = 2\): \(T = (V,E), V = \{u,v\}, E = \{\{u,v\}\}. \) Here \(\Delta = 1\), and we have 2 leaves, so it checks out as required.

There is only one possible tree when \(n = 3\): \(T = (V,E), V = \{u,v,w\}, E = \{\{u,v\},\{v,w\}\}. \) Here \(\Delta = 2\), and we have 2 leaves, so it checks out as required.

We choose to show two base cases here to avoid a slightly unfortunate edge case in the Induction Step.

Induction Step: (IH) Assume that \(P(k)\) is true, for an arbitrary \(k \in \mathbb{Z}^+, k \geq 2\). Consider an arbitrary tree \(T = (V,E)\) such that \(|V| = k + 1\) and it has maximum degree \(\Delta\). Let \(\ell \in V\) be an arbitrary leaf in \(T\) who has some neighbor \(a\). Consider \(T' = (V',E')\) where \(V' = V \setminus \ell\) and \(E' = E \setminus \{a,\ell\}\).

We know that \(|V'| = k\) and is a tree (since removal of a leaf can never disconnect a tree), so we can apply the Induction Hypothesis on \(T'\).

Note that there are two cases here:

1. \(a\) was the only vertex of degree \(\Delta\) in \(T\).

   It must be the case then that \(a\) has degree \(\Delta - 1\) in \(T'\) and is of maximum degree. The Induction Hypothesis gives us that \(T'\) must have at least \(\Delta - 1\) leaves.

   Further note if \(a\) is a leaf, then it must be the case that \(n = 3\) (convince yourself of this), and that is already shown to be true by the base case. Hence, going forward we will operate under the assumption that \(a\) is not a leaf.

   Adding \(\ell\) back to \(T'\) to reconstruct \(T\) increases the number of leaves by one (since \(a\) is not a leaf), so we have that \(T\) has at least \(\Delta\) leaves.

2. There is some vertex in \(T'\) that has degree \(\Delta\).

   By the Induction Hypothesis, we have that \(T'\) must have \(\Delta\) leaves.

   There are two more cases here:

   (a) \(a\) is a leaf in \(T'\)

   In this case, the addition of \(\ell\) does not change the number of leaves, which means we have at least \(\Delta\) leaves in \(T\), as desired.

   (b) \(a\) is not a leaf in \(T'\)

   In this case, the addition of \(\ell\) increases the number of leaves by 1, which means we have at least \(\Delta + 1\) leaves in \(T\), which proves our claim.

Induction on the number of edges:

You can do a similar procedure to the induction on the number of vertices in order to perform induction on the number of edges. Note that in this case you would consider the subgraph induced by the vertices other than the leaf.

Strong Induction on the number of edges:

Let us prove this by induction on the number of edges in the graph \(m\).
We formulate a proposition $P(m)$ which is: in a tree with $m$ edges and maximum degree $\Delta$, the number of leaves in the tree is at least $\Delta$.

**Base Case ($m=1$ and $2$):** There is only one possible tree when $m = 1$: $T = (V, E)$, $V = \{u, v\}$, $E = \{\{u, v\}\}$. Here $\Delta = 1$, and we have 2 leaves, so it checks out as required.

There is only one possible tree when $m = 2$: $T = (V, E)$, $V = \{u, v, w\}$, $E = \{\{u, v\}, \{v, w\}\}$. Here $\Delta = 2$, and we have 2 leaves, so it checks out as required.

We choose to show two base cases here to avoid a slightly unfortunate edge case in the Induction Step.

**Induction Hypothesis:** Assume that $P(j)$ is true, for $j \in \mathbb{Z}, 1 \leq j \leq k$, for an arbitrary $k \in \mathbb{Z}^+$. 

**Induction Step:** Let $T$ be a tree with $k + 1$ edges and with a maximum degree $\Delta$. Let $v$ be a vertex with degree $\Delta$, and $u$ be an arbitrary neighbor of $v$. Let us consider $G' = (V', E')$, where $V' = V$, $E' = E \setminus \{\{u, v\}\}$. Note that $G'$ must have had two connected components $C_1$ and $C_2$, which are both trees when a subgraph is induced on each of them. Let $C_1$ be the component with $v$, and let $C_2$ be the component with $u$.

There are two cases here:

1. There is another vertex in $C_1$ that have degree $\Delta$

   From the induction hypothesis, we have that there must be $\Delta$ leaves in $C_1$. Let us reconstruct $T$ from $G'$.

   There are two cases here:
   
   (a) $|C_2| = 1$

   In this case, if $v$ is a leaf in $G'$, then the addition of $\{u, v\}$ will not change the number of leaves. Therefore we have that $T$ must have at least $\Delta$ leaves. If $v$ is not a leaf, then the addition of $\{u, v\}$ will add an additional leaf, so we have that $T$ must have at least $\Delta + 1$ leaves.

   (b) $|C_2| \geq 2$

   In this case, $C_2$ must have two leaves. Hence there are at least $\Delta + 2$ leaves in $G'$. Notice that the addition of the edge $\{u, v\}$ can decrease the number of leaves by up to 2 (if $u$ and $v$ were both leaves in $G'$). Hence we have that $T$ has at least $\Delta$ leaves, as required.

2. $v$ is the only vertex with degree $\Delta$ in $T$.

   Hence, $\Delta(C_1) = \Delta - 1$. From the induction hypothesis, we know that $C_1$ must have $\Delta - 1$ leaves. We further note that if $v$ is a leaf in $G'$, it must be that $m = 2$ (convince yourself of this), and we have already shown the validity of this in the base case. We will therefore operate now under the assumption that $v$ is not a leaf.

   There are two cases here:
   
   (a) $|C_2| = 1$

   Since $v$ is not a leaf, then the addition of $\{u, v\}$ will add an additional leaf, so we have that $T$ must have at least $\Delta$ leaves.
(b) \(|C_2| \geq 2\)

In this case, \(C_2\) must have two leaves. Hence there are at least \(\Delta + 1\) leaves in \(G'\). Notice that the addition of the edge \(\{u, v\}\) can decrease the number of leaves by up to 1 (if \(u\) is a leaf in \(G'\)). Hence we have that \(T\) has at least \(\Delta\) leaves, as required.

**Using inequalities:**

We know that a tree with \(n\) vertices must have \(n - 1\) edges. Since the sum of the degrees of all the vertices in a graph must be twice the number of edges, we know that the total of all degrees in the tree must be \(2n - 2\).

Let us consider the following partitioning of the vertices in \(V\). Let \(A = \{v \in V \mid \deg(v) = \Delta\}\), \(B = \{v \in V \mid 1 < \deg(v) < \Delta\}\), and \(C = \{v \in V \mid \deg(v) = 1\}\). Note that \(V = A \cup B \cup C\) and \(A \cap B = \emptyset, A \cap C = \emptyset, B \cap C = \emptyset\). Note that \(C\) is the set of leaves.

\[
2n - 2 = \sum_{v \in V} \deg(v) = \sum_{v \in A} \deg(v) + \sum_{v \in B} \deg(v) + \sum_{v \in C} \deg(v) = \Delta \cdot |A| + \sum_{v \in B} \deg(v) + |C| \geq \Delta \cdot |A| + |C| + 2 \cdot |B|
\]

\[
= \Delta \cdot |A| + |C| + 2 \cdot (n - |A| - |C|) = (\Delta - 2) \cdot |A| - |C| + 2n \geq (\Delta - 2) - |C| + 2n
\]

Hence we have established that \(2n - 2 \geq (\Delta - 2) - |C| + 2n\). Further, we have that:

\[
2n - 2 \geq (\Delta - 2) - |C| + 2n
\]

\[
-2 \geq \Delta - 2 - |C| \]

\[
|C| \geq \Delta
\]

Hence we have that the number of leaves is at least \(\Delta\).

**Minimal Counterexample:**

Consider a minimal counterexample, i.e. a tree \(T\) which violates this property with the minimum possible number of vertices, say \(m\). We know that the case for \(m = 1, 2\) can be handled easily, so we may assume that \(m \geq 3\), i.e. the tree has at least 3 vertices. Now pick an arbitrary leaf \(\ell\) and name its only neighbor in the graph \(v\); remove \(\ell\). Consider the resulting graph \(T'\). Note that \(T'\) has exactly \(m - 1\) vertices. The following cases can occur:

*Case 1: \(\Delta(T) = \Delta(T')\).*

Note that by removing a single leaf, we can never increase the number of leaves in the graph. It follows that \(T'\) has at most as many leaves as \(T\), i.e.

\[
\lambda(T') \leq \lambda(T) < \Delta(T) = \Delta(T')
\]
But this means $\lambda(T') < \Delta(T')$ and $T'$ has $m - 1$ vertices. This is a contradiction, as we chose $T$ to be tree with the fewest vertices which violates the claim.

**Case 2:** $v$ is a leaf in $T'$ and $\Delta(T) \neq \Delta(T')$.

The only vertex whose degree can be affected by removing $\ell$ is $v$. Then $v$ must have degree 1 and all other vertices must have degree $\leq 1$. The only trees for which this hold have exactly 1 or 2 vertices; we already know that these cases do not violate the claim. As such, we have a contradiction (it’s impossible for us to end up in this scenario).

**Case 3:** $v$ is not a leaf in $T'$ and $\Delta(T) \neq \Delta(T')$. If the maximum degree changes by removing this leaf, that means that it must decrease by exactly one (we cannot increase degree by removing edges and only removed one edge). In other words, $\Delta(T') = \Delta(T) - 1$. Note that the number of leaves in $T'$ is $\lambda(T) - 1$, since we removed $\ell$ and $v$ is not a leaf. It follows that

$$\lambda(T') = \lambda(T) - 1 < \Delta(T) - 1 = \Delta(T')$$

Again, $\lambda(T') < \Delta(T')$ and we have a contradiction, since $T$ is not minimal.

In every case we have a contradiction - it must be the case that the set of counterexample is empty, i.e. there are no trees which violate the claim, and we are finished.
Problem 2:

In this problem we illustrate a common trap that we can fall in when proving statements about graphs by induction on the number of vertices or the number of edges. Here is a false statement: “if every vertex in a graph has strictly positive (> 0) degree, then the graph is connected”.

(a) Prove that the statement is indeed false by providing a counterexample.

(b) Since the statement is false, there must be a bug in the following “proof”. Pinpoint the first logical mistake (unjustified step).

**Buggy Proof** We prove the statement by induction on the number of vertices. Let \( P(n) \) be the predicate: “for any graph with \( n \) vertices, if every vertex has strictly positive degree, then the graph is connected”.

**Base Cases** \( P(1) \) is vacuously true (this is not where the bug is). Base case \( n = 2 \): there is only one graph with two vertices of strictly positive degree, namely, the graph with an edge between the vertices, and this graph is connected.

**Inductive Step** Let \( k \) be an arbitrary integer such that \( k \geq 2 \). (The assumption \( k \geq 2 \) is OK, this is not where the bug is. We can make such an assumption: the base case of the induction is really \( P(2) \). In such a proof, \( P(1) \) is proved separately. For brevity we included it among the base cases above.).

Assume (IH) \( P(k) \). We must show that this implies \( P(k + 1) \).

Consider a graph \( G_{old} \) with \( k \) vertices in which every vertex has strictly positive degree. By the Induction Hypothesis this graph is connected. Now we add one more vertex, call it \( u \), to obtain a graph \( G_{new} \) with \( k + 1 \) vertices.

All that remains is to check that in \( G_{new} \) there is a walk from \( u \) to every other vertex \( v \). Since \( u \) has positive degree, there is an edge from \( u \) to some other vertex, say \( w \). But \( w \) and \( v \) are in \( G_{old} \), which is connected, and therefore there is a walk from \( w \) to \( v \). This gives a walk \( u \rightarrow w \rightarrow v \) in \( G_{new} \). Done.

**Solution:**

(a) Consider the graph \( G = (V, E) \) where \( V = \{a, b, c, d\} \) and \( E = \{\{a, b\}, \{c, d\}\} \). Every vertex has degree one, however the graph is not connected (there is no path from \( a \) to \( c \), for example).

(b) The bug is in the step “now we add one more vertex”. It is certainly possible to add one more vertex \( u \) such that that it will have strictly positive degree (which is then used in the proof), for example add also an edge. But this constructs a particular graph \( G_{new} \) with \( k + 1 \) vertices. In fact, \( P(k + 1) \) states “for any graph with \( k + 1 \) vertices, etc” so whatever we are proving, it is not \( P(k + 1) \), because we haven’t shown the property for any graph. A correct solution would start with an arbitrary graph with \( k + 1 \) vertices, then construct a graph with \( k \) vertices to which we would apply the IH.

*Note:* Some may argue that this process actually does generate all graphs on \( k + 1 \) vertices, where each vertex has positive degree. However, in order for this to be true, we see that we would need to be able to generate all graphs on \( k + 1 \) vertices where each vertex has positive degree from some graph on \( k \) vertices where each vertex has positive degree. More formally, define \( Q(n) \) to be:
All graphs $G$ on $n$ vertices, where each vertex has positive degree, can be formed by adding a vertex and some non-zero number edges to some graph with $n-1$ vertices, say $G'$, where each vertex in $G'$ has positive degree.

If we can show that $\forall n, Q(n)$ is true, where $n \in \mathbb{Z}^+$, then we know that our process in the Induction Step above would in fact prove $P(k+1)$. However, this claim is actually not true. Consider the example graph in part (a) above. Clearly, removing any vertex from this graph leaves us with a graph where not all of the vertices have positive degree. Thus, we are not able to argue that our Induction Step above proves $P(k+1)$. 


Problem 3:

Consider a connected graph $G = (V, E)$ and an arbitrary partition of $G$’s vertex set into nonempty sets $S$ and $V \setminus S$. Prove that if there exists only one edge $e$ between the vertices in $S$ and the vertices in $V \setminus S$, then $e$ must be in every spanning tree of $G$.

Solution:

Consider an arbitrary spanning tree of $G$, say $T$. Because $T$ is a tree, we know that it is connected, and thus there is a path between any pair of vertices.

Consider a vertex $x \in S$ and consider another vertex $y \in V \setminus S$. Because $T$ is connected, there must be a path $P$ from $x$ to $y$ in $T$. Let us consider this path.

We know that $P$ goes from a vertex in $S$ to a vertex in $V \setminus S$. Therefore, there must exist an edge in the path that crosses the cut between $S$ and $V \setminus S$ (if not, then the path would always stay in either $S$ or $V \setminus S$, which it clearly doesn’t). However, we know that the only edge crossing the cut between $S$ and $V \setminus S$ is $e$. Therefore, our tree $T$ must contain $e$. 
Problem 4:
Consider an undirected graph \( G = (V, E) \) and a rooted spanning tree \( T \) with root \( u \). Consider the layers of \( T \), which can be denoted as \( \{u\} = l_0, l_1, \ldots, l_k \). Layers are defined as the set of vertices that are the same distance from \( u \). Prove that \( G \) is bipartite iff there are no edges in \( G \) between the vertices that exist in layers enumerated with the same parity (for example, no edge between \( l_2 \) and \( l_6 \) and no edge between \( l_3 \) and \( l_3 \)).

Solution:

( \( \implies \) ): We first prove the forward direction, namely that if \( G \) is a bipartite graph, then there are no edges in \( G \) between vertices that exist in the layers enumerated with the same parity. Consider a valid coloring of \( G \); assume WLOG that \( u \) is red. We claim that the layers \( l_0, l_1, \ldots, l_k \) must alternate between red and blue. Since every vertex in \( l_0 \) (only \( u \)) is red, and every vertex in \( l_1 \) has an edge to its parent in \( l_0 \), all vertices in \( l_1 \) are blue. Similarly, every vertex in \( l_2 \) must be red. We can show that this pattern continues (every vertex in \( l_i \) has to have a different color from its parent in \( l_{i-1} \)), so the red vertices occur precisely in the even layers, and the blue vertices occur precisely in the odd layers.

But we know that no edge can have both endpoints of the same color. Hence, it is impossible to have an edge between two layers of the same parity.

( \( \impliedby \) ): We now prove the reverse direction, namely that if \( G \) is a graph where there are no edges between the vertices that exist in the layers enumerated with the same parity, then \( G \) is bipartite. In order to show that \( G \) is bipartite, we will show that it is 2-colorable.

We construct a 2-coloring of the vertices of \( G \) as follows: color all vertices in even-numbered layers \((l_0, l_2, \ldots)\) red and all vertices in odd-numbered layers \((l_1, l_3, \ldots)\) blue. We now show that this 2-coloring is valid, thus showing that \( G \) is 2-colorable and bipartite. Consider any edge \( \{s, t\} \in E \). Since no edge in \( G \) exists between vertices in layers enumerated with the same parity, we know that one of \( s \) and \( t \) belongs to an even-numbered layer, while the other belongs to an odd-numbered layer. Then we know that they are not the same color, thus showing our coloring is valid and proving our claim.