Automata, Computability and Complexity
Jean Gallier

Homework 1

January 13, 2016; Due January 25, 2016, beginning of class

“A problems” are for practice only, and should not be turned in.

Problem A1. Given an alphabet Σ, prove that the relation ≤₁ over Σ* defined such that u ≤₁ v iff u is a prefix of v, is a partial ordering. Prove that the relation ≤₂ over Σ* defined such that u ≤₂ v iff u is a suffix of v, is a partial ordering. Prove that the relation ≤₃ over Σ* defined such that u ≤₃ v iff u is a substring of v, is a partial ordering.

Problem A2. Let Σ be an alphabet, for any languages L₁, L₂, L₃ ⊆ Σ*, prove that if L₁ ⊆ L₂, then L₁L₃ ⊆ L₂L₃.

Problem A3. Let Σ be an alphabet. Given any two families of languages (Aᵢ)ᵢ∈I and (Bⱼ)ⱼ∈J, where I and J are any arbitrary index sets and Aᵢ, Bⱼ ⊆ Σ*, prove that

\[( \bigcup_{i \in I} A_i ) ( \bigcup_{j \in J} B_j ) = \bigcup_{(i,j) \in I \times J} A_i B_j.\]

“B problems” must be turned in.

Problem B1 (20 pts). Let Σ be any alphabet. Given a string w ∈ Σ*, its reversal wᴿ is defined inductively as follows: ϵᴿ = ϵ and (ua)ᴿ = auᴿ, where a ∈ Σ and u ∈ Σ*.

1. Prove that (uv)ᴿ = vᴿuᴿ.
2. Prove that (wᴿ)ᴿ = w.

Problem B2 (40 pts). Given any alphabet Σ, prove the following property: for any two strings u, v ∈ Σ*, uv = vu iff there is some w ∈ Σ* such that u = wᵐ and v = wⁿ, for some m, n ≥ 0.

Hint. In the “hard” direction, consider the subcases

1. |u| = |v|,
2. |u| < |v| and
3. |u| > |v|
and use an induction on $|u| + |v|$.

**Problem B3 (30 pts).**

1. Given any alphabet $\Sigma$, prove that the lexicographic ordering $\preceq$ on $\Sigma^*$ (defined in the slides) is a partial order.

2. Prove that the lexicographic ordering is a total order, which means that for any two strings $u, v \in \Sigma^*$, either $u \preceq v$ or $v \preceq u$.

3. If $\Sigma = \{a, b\}$, give an infinite sequence $(u_i)$ of distinct strings $u_i \in \{a, b\}^*$ such that

   $$\cdots \preceq u_{i+1} \preceq u_i \preceq \cdots \preceq u_1 \preceq u_0$$

   for all $i \in \mathbb{N}$.

**Problem B4 (20 pts).** Given an alphabet $\Sigma$, for any language $L \subseteq \Sigma^*$, prove that $L^* L^* = L^*$ and $L^{**} = L^*$.

*Hint.* To prove that $L^{**} = L^*$, prove that $(L^*)^n = L^*$ for all $n \geq 1$.

**Problem B5 (30 pts).** Let $L$ be any language over some alphabet $\Sigma$.

1. Prove that $L = L^+$ iff $LL \subseteq L$.

2. Prove that $(L = \emptyset$ or $L = L^*)$ iff $LL = L$.

**TOTAL: 140 points.**