# Spring 2020 CIS 262

# Automata, Computability and Complexity
## Jean Gallier

# Homework 1

January 16, 2020; Due January 23, 2020, beginning of class

Mostly a review of induction.

**Problem B1 (50 pts).** Let $S\colon \mathbb{N} \to \mathbb{N}$ be the function given by

$$S(n) = n + 1, \quad \text{for all} \in \mathbb{N}.$$

Define the function $add\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ recursively as follows for all $m, n \in \mathbb{N}$:

$$add(m, 0) = m \qquad \text{(A)}$$
$$add(m, S(n)) = S(add(m, n)). \qquad \text{(B)}$$

(1) Prove that

$$add(add(m, n), p) = add(m, add(n, p))$$

for all $m, n, p \in \mathbb{N}$. In other words, $add$ is associative.
*Hint*. Use induction on $p$. It turns out that $add(m, n) = m+n$, where $+$ is the usual addition of natural numbers but **you can't use this fact!**

(2) We would like to prove that

$$add(m, n) = add(n, m), \quad \text{for all } m, n \in \mathbb{N}$$

but this is a little tricky. First prove

(2a)

$$add(0, n) = n, \quad \text{for all } n \in \mathbb{N}.$$

Also prove

(2b)

$$add(S(m), n) = S(add(m, n)), \quad \text{for all } m, n \in \mathbb{N}.$$

*Hint*. Use induction on $n$.

Finally, prove that

$$add(m, n) = add(n, m), \quad \text{for all } m, n \in \mathbb{N}.$$

In other words, *add* is commutative.
*Hint*. Use induction on $m$.

**Problem B2 (10 pts).** Let $\Sigma$ be any alphabet. For any string $w \in \Sigma^*$ recall that $w^n$ is defined inductively as follows:

$$\begin{aligned} w^0 &= \epsilon \\ w^{n+1} &= w^n w, \quad n \in \mathbb{N}. \end{aligned}$$

For any string $w \in \Sigma^*$ and any natural numbers $m, n \in \mathbb{N}$, prove that

$$w^m w^n = w^{m+n}.$$

*Hint*: Use induction on $n$.

**Problem B3 (40 pts).** Let $\Sigma$ be any alphabet. Given a string $w \in \Sigma^*$, its reversal $w^R$ is defined inductively as follows: $\epsilon^R = \epsilon$, and $(ua)^R = au^R$, where $a \in \Sigma$ and $u \in \Sigma^*$.

(1) Prove that $(uv)^R = v^R u^R$, for all $u, v \in \Sigma^*$.

(2) Prove that $(w^R)^R = w$, for all $w \in \Sigma^*$.

**TOTAL: 100 points.**