Problem B1 (30 pts). Given any set, $X$, for any subset, $A \subseteq X$, recall that the characteristic function, $\chi_A$, of $A$ is the function defined so that

$$\chi_A(x) = \begin{cases} 1 & \text{iff } x \in A \\ 0 & \text{iff } x \in X - A. \end{cases}$$

(i) Prove that, for any two subsets, $A, B \subseteq X$,

$$\chi_{A \cap B} = \chi_A \cdot \chi_B$$

$$\chi_{A \cup B} = \chi_A + \chi_B - \chi_A \cdot \chi_B.$$

(ii) Prove that the union and the intersection of any two r.e. sets, $A, B \subseteq \mathbb{N}$, is also an r.e. set. Prove that the union and the intersection of any two recursive sets, $A, B \subseteq \mathbb{N}$, is also a recursive set.

(Extra Credit (30 pts). Given any $n \geq 2$ subsets, $A_1, A_2, \ldots, A_n \subseteq X$, prove that

$$\chi_{A_1 \cap \cdots \cap A_n} = \chi_{A_1} \cdot \cdots \cdot \chi_{A_n}$$

$$\chi_{A_1 \cup \cdots \cup A_n} = \sum_{I \subseteq \{1, \ldots, n\}, I \neq \emptyset} (-1)^{|I| - 1} \prod_{i \in I} \chi_{A_i}$$

Problem B2 (40 pts). Consider the Post Correspondence Problem in the case where the alphabet $\Sigma$ has a single letter $a$.

(1) Consider the instance of the PCP given by $U = (a^2, a^4, a^5, a^{11}, a^{14}, a^{20})$ and $V = (a^3, a^5, a^6, a^{13}, a^{17}, a^{23})$. Prove that this problem has no solution.

(2) Consider the instance of the PCP given by $U = (a^2, a^5, a^1, a^3, a^4, a^7)$ and $V = (a^{20}, a^2, a^6, a^{26}, a^{37}, a^{23})$. Prove that this problem has a solution and find a solution whose length is 28.

(3) If $U = (a^{p_1}, \ldots, a^{p_m})$ and $V = (a^{q_1}, \ldots, a^{q_m})$ and $p_i = q_i$ for some $i$, then the PCP has the trivial solution for the sequence $(i)$, so assume that $p_i \neq q_i$ for $i = 1, \ldots, m$. In this
case, prove that the PCP has a solution iff there exist some \( i, j \) with \( 1 \leq i, j \leq m \) and \( i \neq j \) such that \( p_i > q_i \) and \( p_j < q_j \).

Problem B3 (20 pts). Prove that the following properties of partial recursive functions are undecidable:

(a) A partial recursive function is a constant function.

(b) Two partial recursive functions \( \varphi_x \) and \( \varphi_y \) are identical.

(c) A partial recursive function \( \varphi_x \) is equal to a given partial recursive function \( \varphi_a \).

(d) A partial recursive function diverges for all input.

Problem B4 (50 pts). Given an undirected graph \( G = (V, E) \) and a set \( C = \{c_1, \ldots, c_p\} \) of \( p \) colors, a coloring of \( G \) is an assignment of a color from \( C \) to each node in \( V \) such that no two adjacent nodes share the same color, or more precisely such that for every edge \( \{u, v\} \in E \), the nodes \( u \) and \( v \) are assigned different colors. A \( k \)-coloring of a graph \( G \) is a coloring using at most \( k \)-distinct colors. For example, the graph shown in Figure 1 has a 3-coloring (using green, blue, red).

![Figure 1: Petersen graph.](image)

The graph coloring problem is to decide whether a graph \( G \) is \( k \)-colorable for a given integer \( k \geq 1 \).
(1) Give a polynomial reduction from the graph coloring problem to the satisfiability problem for propositions in CNF.

If $|V| = n$, create $n \times k$ propositional variables $x_{ij}$ with the intended meaning that $x_{ij}$ is true iff node $v_i$ is colored with color $j$. You need to write sets of clauses to assert the following facts:

1. Every node is colored.
2. No two distinct colors are assigned to the same node.
3. For every edge $\{v_i, v_j\}$, nodes $v_i$ and $v_j$ cannot be assigned the same color.

Beware that it is possible to assert that every node is assigned one and only one color using a proposition in disjunctive normal form, but this is not a correct answer; we want a proposition in conjunctive normal form.

(2) Use the above reduction to prove that 2-coloring can be solved deterministically in polynomial time,

Remark: It is known that a graph has a 2-coloring iff its is bipartite. The problem of 3-coloring is actually $\mathcal{NP}$-complete but this is a bit tricky to prove.

(Extra Credit (30 pts).) Prove that Petersen graph has no Hamiltonian cycle.

Problem B5 (60 pts). Let $A$ be any $p \times q$ matrix with integer coefficients and let $b \in \mathbb{Z}^p$ be any vector with integer coefficients. The 0-1 integer programming problem is to find whether a system of $p$ linear equations in $q$ variables

$$a_{11}x_1 + \cdots + a_{1q}x_q = b_1$$
$$\vdots$$
$$a_{i1}x_1 + \cdots + a_{iq}x_q = b_i$$
$$\vdots$$
$$a_{pq}x_1 + \cdots + a_{pq}x_q = b_p$$

with $a_{ij}, b_i \in \mathbb{Z}$ has any solution $x \in \{0, 1\}^q$, that is, with $x_i \in \{0, 1\}$. In matrix form, if we let

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1q} \\
\vdots & \ddots & \vdots \\
a_{p1} & \cdots & a_{pq} \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\
\vdots \\
b_p \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\
\vdots \\
x_q \end{pmatrix},$$

then we write the above system as

$$Ax = b.$$
(i) Prove that the 0-1 integer programming problem is in \( \mathcal{NP} \).

(ii) Prove that the restricted 0-1 integer programming problem in which the coefficients of \( A \) are 0 or 1 and all entries in \( b \) are equal to 1 is \( \mathcal{NP} \)-complete by providing a polynomial-time reduction from the bounded-tiling problem. Do not try to reduce any other problem to the 0-1 integer programming problem.

Hint. Given a tiling problem, \( ((T, V, H), \tilde{s}, \sigma_0) \), create a 0-1-valued variable, \( x_{mnt} \), such that \( x_{mnt} = 1 \) iff tile \( t \) occurs in position \( (m, n) \) in some tiling. Write equations or inequalities expressing that a tiling exists and then use “slack variables” to convert inequalities to equations. For example, to express the fact that every position is tiled by a single tile, use the equation

\[
\sum_{t \in T} x_{mnt} = 1,
\]

for all \( m, n \) with \( 1 \leq m \leq 2s \) and \( 1 \leq n \leq s \). Also, if you have an inequality such as

\[
2x_1 + 3x_2 - x_3 \leq 5 \quad (\ast)
\]

with \( x_1, x_2, x_3 \in \mathbb{Z} \), then using a new variable \( y_1 \) taking its values in \( \mathbb{N} \), that is, nonnegative values, we obtain the equation

\[
2x_1 + 3x_2 - x_5 + y_1 = 5, \quad (\ast\ast)
\]

and the inequality (\( \ast \)) has solutions with \( x_1, x_2, x_3 \in \mathbb{Z} \) iff the equation (\( \ast\ast \)) has a solution with \( x_1, x_2, x_3 \in \mathbb{Z} \) and \( y_1 \in \mathbb{N} \). The variable \( y_1 \) is called a slack variable (this terminology comes from optimization theory, more specifically, linear programming). For the 0-1-integer programming problem, all variables, including the slack variables, take values in \( \{0, 1\} \).

Conclude that the 0-1 integer programming problem is \( \mathcal{NP} \)-complete.

TOTAL: 200 points + 60 points Extra credit