Spring 2020 CIS 262

Automata, Computability and Complexity
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Homework 2

January 23, 2020; Due January 30, 2020, beginning of class

Problem B1 (70 pts). Let $\Sigma = \{a_1, \ldots, a_k\}$ be any alphabet. Given a string $w \in \Sigma^*$, its reversal w^R is defined inductively as follows: $\epsilon^R = \epsilon$, and $(ua)^R = au^R$, where $a \in \Sigma$ and $u \in \Sigma^*$.

A *palindrome* is a string w such that $w = w^R$. Here are some examples of palindromes:

eye
racecar
never odd or even
god saw I was dog
campus motto bottoms up mac
do geese see god

If $k = 1$, every string is a palindrome. Therefore we assume that $k \geq 2$.

We would like to give a formula giving the number p_n of all palindromes w of length $|w| = n \geq 0$ over the alphabet $\Sigma = \{a_1, \ldots, a_k\}$ with k letters.

(1) Prove that a palindrome $w \in \Sigma^*$ is either the empty string $w = \epsilon$, or $w = a$ with $a \in \Sigma$, or $w = aua$ where u is a palindrome of length $n - 2$ where $n = |w| \geq 2$ and $a \in \Sigma$ is some letter.

(2) Prove that $p_0 = 1$, $p_1 = k$, and

$$p_{n+2} = kp_n, \quad \text{for all } n \geq 0.$$

Give a formula for p_n. Distinguish between the cases where $n = 2m$ (n is even) and $n = 2m + 1$ (n is odd). You must prove the correctness of your formulae (use induction).

Do **not** give formulae in terms of $n/2$ when n is even or $(n - 1)/2$ when n odd. Please give formulae for p_{2m} and p_{2m+1} in terms of m.

(3) Prove that the number P_n of all palindromes w of length $\leq n$ (which means that $0 \leq |w| \leq n$) over the alphabet $\Sigma = \{a_1, \ldots, a_k\}$ with k letters is given by

$$P_{2m} = \frac{2k^{m+1} - k - 1}{k - 1} \qquad n = 2m$$
$$P_{2m+1} = \frac{k^{m+2} + k^{m+1} - k - 1}{k - 1} \qquad n = 2m + 1,$$

for any natural number $m \in \mathbb{N}$. Prove that the number Q_n of all non-palindromes w of length $\leq n$ over the alphabet $\Sigma = \{a_1, \ldots, a_k\}$ is given by

$$Q_{2m} = \frac{k^{2m+1} - 2k^{m+1} + k}{k - 1} \qquad n = 2m$$
$$Q_{2m+1} = \frac{k^{2m+2} - k^{m+2} - k^{m+1} + k}{k - 1} \qquad n = 2m + 1,$$

for any natural number $m \in \mathbb{N}$.
Hint. Figure out the total number of strings of length $\leq n$ over an alphabet of size $k \geq 2$.

(4) If $k = 2$, prove that if $m \geq 2$, then $P_{2m}/Q_{2m} < 1$ and $P_{2m+1}/Q_{2m+1} < 1$, so there are more non-palindromes than palindromes. What is 536 870 909 (in relation to palindromes)? Show that

$$\frac{536\,870\,909}{2^{55} - 1} \approx 2^{-26} \approx 1.4901 \times 10^{-8}.$$

What the interpretation of the above ratio as a probability?

Problem B2 (30 pts). Let Σ be any alphabet. For any string $w \in \Sigma^*$ recall that w^n is defined inductively as follows:

$$w^0 = \epsilon$$
$$w^{n+1} = w^n w, \quad n \in \mathbb{N}.$$

Prove the following property: for any two strings $u, v \in \Sigma^*$, $uv = vu$ iff there is some $w \in \Sigma^*$ such that $u = w^m$ and $v = w^n$, for some $m, n \geq 0$.

Hint. In the "hard" direction, consider the subcases

(1) $|u| = |v|$,

(2) $|u| < |v|$ and

(3) $|u| > |v|$

and use an induction on $|u| + |v|$.

TOTAL: 100 points.