Problem B1 (80 pts). Let $\Sigma = \{a_1, \ldots, a_k\}$ be any alphabet with $k \geq 2$. Assume that $\Sigma$ is totally ordered so that $a_1 < a_2 < \cdots < a_k$ and let $\preceq$ be the corresponding lexicographic ordering. Define the binary relation $\ll$ on $\Sigma^*$ as follows: for all $u, v \in \Sigma^*$

$u \ll v \begin{cases} (1) & \text{if } |u| < |v|, \text{ or} \\ (2) & |u| = |v| \text{ and } u \preceq v. \end{cases}$

(1) Prove that $\ll$ is partial order which is also a total order.

(2) Define the function $\nu: \Sigma^* \to \mathbb{N}$ as follows:

$\nu(\epsilon) = 0,$

and for any nonempty string $u = a_{i_1} \cdots a_{i_n}$ (with $i_j \in \{1, \ldots, k\}, j = 1, \ldots n),$

$\nu(u) = i_1 k^{n-1} + i_2 k^{n-2} + \cdots + i_{n-1} k + i_n.$

Prove that for all $u, v \in \Sigma^*$,

(i) If $|u| < |v|$ then $\nu(u) < \nu(v)$.

(ii) If $|u| = |v|$, $u = a_i x$, and $v = a_j y$ with $i < j$, then $\nu(u) < \nu(v)$.

(3) Prove that $\nu$ is strictly monotonic with respect to $\ll$, which means that

if $u \ll v$ and $u \neq v$ then $\nu(u) < \nu(v)$.

Use this to prove $\nu: \Sigma^* \to \mathbb{N}$ is injective.

(4) Prove that $\nu: \Sigma^* \to \mathbb{N}$ is surjective.

Hint. Use complete induction on $m$. When $m > 0$ divide $m$ by $k$, so that $m = kq + r$ with $0 \leq r \leq k - 1$. The case where $r = 0$ requires special handling.
Problem B2 (50 pts). Let $\Sigma = \{a_1, \ldots, a_n\}$ be an alphabet of $n$ symbols.

(1) Construct an NFA with $2^n + 1$ states accepting the set $L_n$ of strings over $\Sigma$ such that,
every string in $L_n$ has an odd number of $a_i$, for some $a_i \in \Sigma$. Equivalently, if $L_i^n$ is the set
of all strings over $\Sigma$ with an odd number of $a_i$, then $L_n = L_1^n \cup \cdots \cup L_n^n$.

(2) Prove that there is a DFA with $2^n$ states accepting the language $L_n$.

(3) Prove that every DFA accepting $L_n$ has at least $2^n$ states.

*Hint.* If a DFA $D$ with $k < 2^n$ states accepts $L_n$, show that there are two strings $u, v$ with
the property that, for some $a_i \in \Sigma$, $u$ contains an odd number of $a_i$’s, $v$ contains an even
number of $a_i$’s, and $D$ ends in the same state after processing $u$ and $v$. From this, conclude
that $D$ accepts incorrect strings.

**TOTAL:** 130 points