“A problems” are for practice only, and should not be turned in.

Problem A1. It was shown in class that the union of two regular languages is regular. However, as you will show in this problem, the infinite union of regular languages is not regular, in general.

Consider the language \( \{a^n b^n \mid n \geq 1 \} \), which is known not to be regular (we will prove this in class, don’t worry about proving it for this problem).

Find an infinite family of languages, \( \{L_i\}_{i \geq 1} \), with every \( L_i \) finite, so that
\[
\{a^n b^n \mid n \geq 1 \} = \bigcup_{i \geq 1} L_i.
\]

What can you conclude about infinite unions of regular languages?

“B problems” must be turned in.

Problem B1 (30 pts). Let \( L \) be any language over some alphabet \( \Sigma \).

(a) Prove that \( L = L^+ \) iff \( LL \subseteq L \).

(b) Prove that \( (L = \emptyset \text{ or } L = L^*) \) iff \( LL = L \).

Problem B2 (50 pts). Let \( D = (Q, \Sigma, \delta, q_0, F) \) be a DFA. Recall that a state \( p \in Q \) is accessible or reachable iff there is some string \( w \in \Sigma^* \), such that
\[
\delta^*(q_0, w) = p,
\]
i.e., there is some path from \( q_0 \) to \( p \) in \( D \). Consider the following method for computing the set \( Q_r \) of reachable states (of \( D \)): define the sequence of sets \( Q_r^i \subseteq Q \), where
\[
Q_r^0 = \{q_0\},
\]
\[
Q_r^{i+1} = \{q \in Q \mid \exists p \in Q_r^i, \forall a \in \Sigma, q = \delta(p, a)\}.
\]

(i) Prove by induction on \( i \) that \( Q_r^i \) is the set of all states reachable from \( q_0 \) using paths of length \( i \) (where \( i \) counts the number of edges).
Give an example of a DFA such that $Q_{i+1}^r \neq Q_i^r$ for all $i \geq 0$.

(ii) Give an example of a DFA such that $Q_i^r \neq Q_r$ for all $i \geq 0$.

(iii) Change the inductive definition of $Q_i^r$ as follows:

$$Q_{i+1}^r = Q_i^r \cup \{q \in Q \mid \exists p \in Q_i^r, \exists a \in \Sigma, q = \delta(p, a)\}.$$  

Prove that there is a smallest integer $i_0$ such that

$$Q_{i_0+1}^r = Q_{i_0}^r = Q_r.$$  

Define the DFA $D_r$ as follows: $D_r = (Q_r, \Sigma, \delta_r, q_0, F \cap Q_r)$, where $\delta_r : Q_r \times \Sigma \to Q_r$ is the restriction of $\delta$ to $Q_r$. Explain why $D_r$ is indeed a DFA, and prove that $L(D_r) = L(D)$. A DFA is said to be reachable, or trim, if $D = D_r$.

**Problem B3 (50 pts).** Let $\Sigma = \{a_1, \ldots, a_n\}$ be an alphabet of $n$ symbols.

(1) Construct an NFA with $2n + 1$ states accepting the set $L_n$ of strings over $\Sigma$ such that, every string in $L_n$ has an odd number of $a_i$, for some $a_i \in \Sigma$. Equivalently, if $L_n^i$ is the set of all strings over $\Sigma$ with an odd number of $a_i$, then $L_n = L_n^1 \cup \cdots \cup L_n^n$.

(2) Prove that there is a DFA with $2^n$ states accepting the language $L_n$.

(3) Prove that every DFA accepting $L_n$ has at least $2^n$ states.

*Hint.* If a DFA $D$ with $k < 2^n$ states accepts $L_n$, show that there are two strings $u, v$ with the property that, for some $a_i \in \Sigma$, $u$ contains an odd number of $a_i$'s, $v$ contains an even number of $a_i$’s, and $D$ ends in the same state after processing $u$ and $v$. From this, conclude that $D$ accepts incorrect strings.

**TOTAL: 130 points.**