Spring 2020 CIS 262

Automata, Computability and Complexity
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Homework 3

January 30, 2020; Due February 6, 2020, beginning of class

"A problems" are for practice only, and should not be turned in.

Problem A1. Let Σ be an alphabet, for any languages $L_1, L_2, L_3 \subseteq \Sigma^*$, prove that if $L_1 \subseteq L_2$, then $L_1L_3 \subseteq L_2L_3$.

Problem A2. Let Σ be an alphabet. Given any two families of languages $(A_i)_{i\in I}$ and $(B_j)_{j\in J}$, where I and J are any arbitrary index sets and $A_i, B_j \subseteq \Sigma^*$, prove that

$$\left(\bigcup_{i\in I} A_i\right)\left(\bigcup_{j\in J} B_j\right) = \bigcup_{(i,j)\in I\times J} A_iB_j.$$

"B problems" must be turned in. You may use the solutions to the A problems to solve the B problems.

Problem B1 (30 pts). Given an alphabet Σ, for any language $L \subseteq \Sigma^*$, prove that $L^*L^* = L^*$ and $L^{**} = L^*$.

Hint. To prove that $L^{**} = L^*$, prove that $(L^*)^n = L^*$ for all $n \geq 1$.

Problem B2 (30 pts). Let L be any language over some alphabet Σ.

(a) Prove that $L = L^+$ iff $LL \subseteq L$.

(b) Prove that ($L = \emptyset$ or $L = L^*$) iff $LL = L$.

Problem B3 (40 pts). For any language $L \subseteq \{a\}^*$, prove that if $L = L^*$, then there is a finite language $S \subseteq L$ such that $L = S^*$.

Hint. If $L \neq \{\epsilon\}$, then L contains some nonempty strings, and there is a shortest nonempty string $a^m \in L$. Consider the finite set S of strings in L of the form a^{mq+r}, where $0 \leq r \leq m-1$, and where $q \geq 0$ is minimal.

Problem B4 (20 pts). For any set X and *any* function $g\colon X \to 2^X$, let D be the set defined in the proof of Cantor's theorem by

$$D = \{x \in X \mid x \notin g(x)\} \in 2^X.$$

Consider the set $X = \{a, b, c\}$.

(1) Give a function $g_1 \colon X \to 2^X$ such that $D = \emptyset$.
(1) Give a function $g_2 \colon X \to 2^X$ such that $D = X = \{a, b, c\}$.

TOTAL: 120 points.