Problem B1 (50 pts). Let $R$ be any regular language over some alphabet $\Sigma$. Prove that the language

$$L^{1/2} = \{ u \in \Sigma^* \mid uu \in R \}$$

is regular.

*Hint.* Let $D = (Q, \Sigma, \delta, q_0, F)$ be a DFA accepting $R$. Express $L^{1/2}$ as a finite union of intersections $L^1_p \cap L^2_p$ of languages $L^1_p$ and $L^2_p$ (with $p \in Q$) accepted by DFA’s obtained by modifying $D$.

Problem B2 (50 pts). (a) Let $T = \{0, 1, 2\}$, let $C$ be the set of 20 strings of length three over the alphabet $T$, 

$$C = \{ u \in T^3 \mid u \notin \{110, 111, 112, 101, 121, 011, 211\} \},$$

let $\Sigma = \{0, 1, 2, c\}$, and consider the language

$$L_M = \{ w \in \Sigma^* \mid w = u_1cu_2c\cdots cu_n, \ n \geq 1, u_i \in C \}.$$ 

Prove that $L_M$ is regular (there is a DFA with 7 states).

(b) The language $L_M$ has a geometric interpretation as a certain subset of $\mathbb{R}^3$ (actually, $\mathbb{Q}^3$), as follows: Given any string, $w = u_1cu_2c\cdots cu_n \in L_M$, denoting the $j$th character in $u_i$ by $u^j_i$, where $j \in \{1, 2, 3\}$, we obtain three strings

$$w^1 = u^1_1u^1_2\cdots u^1_n, \quad w^2 = u^2_1u^2_2\cdots u^2_n, \quad w^3 = u^3_1u^3_2\cdots u^3_n.$$ 

For example, if $w = 012c001c222c122$ we have $w^1 = 0021$, $w^2 = 1022$, and $w^3 = 2122$. Now, a string $v \in T^+$ can be interpreted as a decimal real number written in base three! Indeed, if

$$v = b_1b_2\cdots b_k, \quad \text{where} \quad b_i \in \{0, 1, 2\} = T \ (1 \leq i \leq k),$$
we interpret \( v \) as \( n(v) = 0.b_1b_2 \cdots b_k \), i.e.,

\[
    n(v) = b_13^{-1} + b_23^{-2} + \cdots + b_k3^{-k}.
\]

Finally, a string, \( w = u_1cu_2c \cdots cu_n \in L_M \), is interpreted as the point, \((x_w, y_w, z_w) \in \mathbb{R}^3\), where

\[
    x_w = n(w^1), \quad y_w = n(w^2), \quad z_w = n(w^3).
\]

Therefore, the language, \( L_M \), is the encoding of a set of rational points in \( \mathbb{R}^3 \), call it \( M \). This turns out to be the rational part of a fractal known as the Menger sponge.

Describe recursive rules to create the set \( M \), starting from a unit cube in \( \mathbb{R}^3 \). Justify as best as you can how these rules are derived from the description of the coordinates of the points of \( M \) defined above (which points are omitted, included, ...).

Draw some pictures illustrating this process and showing approximations of the Menger sponge.

**Extra Credit (30 points).** Write a computer program to draw the Menger sponge (based on the ideas above).

**Problem B3 (30 pts).** Consider the following NFA accepting the language \( L = \{aa, aaa\}^* \) (over the alphabet \( \Sigma = \{a\}\)):

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<td>{1}</td>
</tr>
<tr>
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<tr>
<td>2</td>
<td>( \emptyset )</td>
<td>{3}</td>
</tr>
<tr>
<td>3</td>
<td>{0}</td>
<td>( \emptyset )</td>
</tr>
</tbody>
</table>

(1) Apply the subset construction algorithm presented on page 75 of the slides to obtain a DFA for \( L \) (you should get a DFA with 5 states).

(2) The language \( L = \{aa, aaa\}^* \) is of the form \( L = \{a\}^* - S \), where \( S \) is a finite set of strings. What exactly is \( S \)?

(3) Give a DFA with 3 states accepting \( L \).

**TOTAL: 130 points + 30 points extra credit.**