"B problems" must be turned in.

**Problem B1 (80 pts).** Let \(D = (Q, \Sigma, \delta, q_0, F)\) be a deterministic finite automaton. Define the relations \(\approx\) and \(\sim\) on \(\Sigma^*\) as follows:

\[
x \approx y \quad \text{if and only if, for all } p \in Q, \quad \delta^*(p, x) \in F \iff \delta^*(p, y) \in F;
\]

and

\[
x \sim y \quad \text{if and only if, for all } p \in Q, \quad \delta^*(p, x) = \delta^*(p, y).
\]

(a) Show that \(\approx\) is a left-invariant equivalence relation and that \(\sim\) is an equivalence relation that is both left and right invariant. (A relation \(R\) on \(\Sigma^*\) is left invariant iff \(uRv\) implies that \(wuRwv\) for all \(w \in \Sigma^*\), and \(R\) is left and right invariant iff \(uRv\) implies that \(xuyRxvy\) for all \(x, y \in \Sigma^*\).)

(b) Let \(n\) be the number of states in \(Q\) (the set of states of \(D\)). Show that \(\approx\) has at most \(2^n\) equivalence classes and that \(\sim\) has at most \(n^n\) equivalence classes.

*Hint.* In the case of \(\approx\), consider the function \(f: \Sigma^* \to 2^Q\) given by

\[
f(u) = \{p \in Q \mid \delta^*(p, u) \in F\}, \quad u \in \Sigma^*,
\]

and show that \(x \approx y\) iff \(f(x) = f(y)\). In the case of \(\sim\), let \(Q^Q\) be the set of all functions from \(Q\) to \(Q\) and consider the function \(g: \Sigma^* \to Q^Q\) defined such that \(g(u)\) is the function given by

\[
g(u)(p) = \delta^*(p, u), \quad u \in \Sigma^*, \ p \in Q,
\]

and show that \(x \sim y\) iff \(g(x) = g(y)\).

(c) Given any language \(L \subseteq \Sigma^*\), define the relations \(\lambda_L\) and \(\mu_L\) on \(\Sigma^*\) as follows:

\[
u \lambda_L v \quad \text{iff, for all } z \in \Sigma^*, \ zu \in L \iff zv \in L,
\]

and
and
\[ u \mu_L v \text{ iff, for all } x, y \in \Sigma^*, \ xuy \in L \text{ iff } xvy \in L. \]

Prove that \( \lambda_L \) is left-invariant, and that \( \mu_L \) is left and right-invariant. Prove that if \( L \) is regular, then both \( \lambda_L \) and \( \mu_L \) have a finite number of equivalence classes.

Hint: Show that the number of classes of \( \lambda_L \) is at most the number of classes of \( \approx \), and that the number of classes of \( \mu_L \) is at most the number of classes of \( \sim \).

**Problem B2 (80 pts).** This problem illustrates the power of the congruence version of Myhill-Nerode.

Let \( L \) be any regular language over some alphabet \( \Sigma \). Define the languages

\[
L^\infty = \bigcup_{k \geq 1} \{ w^k \mid w \in L \},
\]

\[
L^{1/\infty} = \{ w \mid w^k \in L, \text{ for all } k \geq 1 \}, \text{ and }
\]
\[
\sqrt{L} = \{ w \mid w^k \in L, \text{ for some } k \geq 1 \}.
\]

Also, for any natural number \( k \geq 1 \), let

\[
L^{(k)} = \{ w^k \mid w \in L \},
\]

and

\[
L^{(1/k)} = \{ w \mid w^k \in L \}.
\]

(a) Prove that \( L^{(1/3)} \) is regular. What about \( L^{(3)} \)?

(b) Prove that the languages \( L^{(1/k)} = \{ w \mid w^k \in L \} \) are regular for all \( k \geq 2 \). Prove that there are only finitely many distinct languages of the form \( L^{(1/k)} \), which means that the set of languages \( \{ L^{(1/k)} \mid k \geq 1 \} \) is finite (in fact, if \( L \) is accepted by a DFA with \( n \) states, there are at most \( 2^n \) distinct languages of the form \( L^{(1/k)} \)).

(c) Is \( L^{1/\infty} \) regular or not? Is \( \sqrt{L} \) regular or not? What about \( L^\infty \)?

**TOTAL: 160 points**