“B problems” must be turned in.

**Problem B1 (80 pts).** Let $D = (Q, \Sigma, \delta, q_0, F)$ be a deterministic finite automaton. Define the relations $\approx$ and $\sim$ on $\Sigma^*$ as follows:

$x \approx y$ if and only if, for all $p \in Q$, $
\delta^*(p, x) \in F$ iff $\delta^*(p, y) \in F$,

and

$x \sim y$ if and only if, for all $p \in Q$, $\delta^*(p, x) = \delta^*(p, y)$.

(1) Show that $\approx$ is a left-invariant equivalence relation and that $\sim$ is an equivalence relation that is both left and right invariant. (A relation $R$ on $\Sigma^*$ is left invariant iff $uRv$ implies that $wuRwv$ for all $w \in \Sigma^*$, and $R$ is left and right invariant iff $uRv$ implies that $xuyRxvy$ for all $x, y \in \Sigma^*$.)

(2) Let $n$ be the number of states in $Q$ (the set of states of $D$). Show that $\approx$ has at most $2^n$ equivalence classes and that $\sim$ has at most $n^n$ equivalence classes.

*Hint.* In the case of $\approx$, consider the function $f: \Sigma^* \rightarrow 2^Q$ given by

$f(u) = \{p \in Q \mid \delta^*(p, u) \in F\}, \quad u \in \Sigma^*$,

and show that $x \approx y$ iff $f(x) = f(y)$. In the case of $\sim$, let $Q^Q$ be the set of all functions from $Q$ to $Q$ and consider the function $g: \Sigma^* \rightarrow Q^Q$ defined such that $g(u)$ is the function given by

$g(u)(p) = \delta^*(p, u), \quad u \in \Sigma^*, \quad p \in Q$,

and show that $x \sim y$ iff $g(x) = g(y)$.

(3) Given any language $L \subseteq \Sigma^*$, define the relations $\lambda_L$ and $\mu_L$ on $\Sigma^*$ as follows:

$u \lambda_L v$ iff, for all $z \in \Sigma^*$, $zu \in L$ iff $zv \in L$, 

and $u \mu_L v$ iff, for all $z \in \Sigma^*$, $zu \in L$ iff $zv \in L$. 


and
\[ u \mu_L v \text{ iff, for all } x, y \in \Sigma^*, \ xuy \in L \text{ iff } xvy \in L. \]

Prove that \( \lambda_L \) is left-invariant, and that \( \mu_L \) is left and right-invariant. Prove that if \( L \) is regular, then both \( \lambda_L \) and \( \mu_L \) have a finite number of equivalence classes.

\textit{Hint}: Show that the number of classes of \( \lambda_L \) is at most the number of classes of \( \approx \), and that the number of classes of \( \mu_L \) is at most the number of classes of \( \sim \).

\textbf{Problem B2 (60 pts).} (1) Prove that the intersection, \( L_1 \cap L_2 \), of two regular languages, \( L_1 \) and \( L_2 \), is regular, using the Myhill-Nerode characterization of regular languages.

(2) Let \( h: \Sigma^* \to \Delta^* \) be a homomorphism, as defined on pages 31-33 of the slides on DFA’s and NFA’s. For any regular language, \( L' \subseteq \Delta^* \), prove that
\[ h^{-1}(L') = \{w \in \Sigma^* | h(w) \in L'\} \]
is regular, using the Myhill-Nerode characterization of regular languages.

Proceed as follows: Let \( \simeq' \) be a right-invariant equivalence relation on \( \Delta^* \) of finite index \( n \), such that \( L' \) is the union of some of the equivalence classes of \( \simeq' \). Let \( \simeq \) be the relation on \( \Sigma^* \) defined by
\[ u \simeq v \text{ iff } h(u) \simeq' h(v). \]

Prove that \( \simeq \) is a right-invariant equivalence relation of finite index \( m \), with \( m \leq n \), and that \( h^{-1}(L') \) is the union of equivalence classes of \( \simeq \).

To prove that the index of \( \simeq \) is at most the index of \( \simeq' \), use \( h \) to define a function \( \hat{h}: (\Sigma^* / \simeq) \to (\Delta^* / \simeq') \) from the partition associated with \( \simeq \) to the partition associated with \( \simeq' \), and prove that \( \hat{h} \) is injective.

Prove that the number of states of any minimal DFA for \( h^{-1}(L') \) is at most the number of states of any minimal DFA for \( L' \). Can it be strictly smaller? If so, give an explicit example.

\textbf{TOTAL: 140 points}