Spring 2020 CIS 262

Automata, Computability and Complexity
Jean Gallier

Homework 7

March 26, 2020; Due April 7, 2020, beginning of class

"B problems" must be turned in.

Problem B1 (80 pts). Let $D = (Q, \Sigma, \delta, q_0, F)$ be a deterministic finite automaton. Define the relations $\approx$ and $\sim$ on Σ^* as follows:

$$\begin{aligned} x \approx y \quad &\text{if and only if,} \quad \text{for all} \quad p \in Q, \\ &\delta^*(p, x) \in F \quad \text{iff} \quad \delta^*(p, y) \in F, \end{aligned}$$

and

$$x \sim y \quad \text{if and only if,} \quad \text{for all} \quad p \in Q, \quad \delta^*(p, x) = \delta^*(p, y).$$

(a) Show that $\approx$ is a left-invariant equivalence relation and that $\sim$ is an equivalence relation that is both left and right invariant. (A relation R on Σ^* is *left invariant* iff uRv implies that $wuRwv$ for all $w \in \Sigma^*$, and R is *left and right invariant* iff uRv implies that $xuyRxvy$ for all $x, y \in \Sigma^*$.)

(b) Let n be the number of states in Q (the set of states of D). Show that $\approx$ has at most 2^n equivalence classes and that $\sim$ has at most n^n equivalence classes.

Hint. In the case of $\approx$, consider the function $f\colon \Sigma^* \to 2^Q$ given by

$$f(u) = \{p \in Q \mid \delta^*(p, u) \in F\}, \quad u \in \Sigma^*,$$

and show that $x \approx y$ iff $f(x) = f(y)$. In the case of $\sim$, let Q^Q be the set of all functions from Q to Q and consider the function $g\colon \Sigma^* \to Q^Q$ defined such that $g(u)$ is the function given by

$$g(u)(p) = \delta^*(p, u), \quad u \in \Sigma^*, \ p \in Q,$$

and show that $x \sim y$ iff $g(x) = g(y)$.

(c) Given any language $L \subseteq \Sigma^*$, define the relations λ_L and μ_L on Σ^* as follows:

$$u \, \lambda_L \, v \quad \text{iff,} \quad \text{for all} \quad z \in \Sigma^*, \quad zu \in L \quad \text{iff} \quad zv \in L,$$

and

$$u \, \mu_L \, v \quad \text{iff,} \quad \text{for all} \quad x, y \in \Sigma^*, \quad xuy \in L \quad \text{iff} \quad xvy \in L.$$

Prove that λ_L is left-invariant, and that μ_L is left and right-invariant. Prove that if L is regular, then both λ_L and μ_L have a finite number of equivalence classes.

Hint: Show that the number of classes of λ_L is at most the number of classes of $\approx$, and that the number of classes of μ_L is at most the number of classes of $\sim$.

Problem B2 (80 pts). This problem illustrates the power of the congruence version of Myhill-Nerode.

Let L be any regular language over some alphabet Σ. Define the languages

$$\begin{aligned} L^{\infty} &= \bigcup_{k \geq 1} \{w^k \mid w \in L\}, \\ L^{1/\infty} &= \{w \mid w^k \in L, \quad \text{for all } k \geq 1\}, \quad \text{and} \\ \sqrt{L} &= \{w \mid w^k \in L, \quad \text{for some } k \geq 1\}. \end{aligned}$$

Also, for any natural number $k \geq 1$, let

$$L^{(k)} = \{w^k \mid w \in L\},$$

and

$$L^{(1/k)} = \{w \mid w^k \in L\}.$$

(a) Prove that $L^{(1/3)}$ is regular. What about $L^{(3)}$?

(b) Let $k \geq 1$ be any natural number. Prove that there are only finitely many languages of the form $L^{(1/k)} = \{w \mid w^k \in L\}$ and that they are all regular. (In fact, if L is accepted by a DFA with n states, there are at most $2^{(n^n)}$ languages of the form $L^{(1/k)}$).

(c) Is $L^{1/\infty}$ regular or not? Is $\sqrt{L}$ regular or not? What about L^{∞}?

TOTAL: 160 points