Problem B1 (100 pts). Consider the following language $L$ over \{a, b\}^*:

$L = \{ w \mid \exists n \geq 1, \exists x_i \in a^+, \exists y_i \in b^+, 1 \leq i \leq n, n \text{ is not prime}, w = x_1y_1 \cdots x_ny_n \}.$

(1) Prove that $L$ is not regular.

(2) Prove that there is an integer $m \geq 1$ such that, for every $w \in L$, if $|w| \geq m$, then there is some decomposition of $w$ of the form $w = uxv$ satisfying the following properties:

(a) $x \neq \varepsilon$.

(b) $ux^iv \in L$ for all $i \geq 0$.

(c) $|ux| \leq m$.

In other words, the conclusion of the pumping lemma holds for $L$, yet $L$ is not regular.

Hint. Actually we must have $m \geq 8$. Beware of the case $i = 0$ in (b).

(3) Consider the following version of the pumping lemma. For any regular language $L$, there is some $m \geq 1$ so that for every $y \in \Sigma^*$, if $|y| \geq m$, then there exist $u, x, v \in \Sigma^*$ so that

(1) $y = uxv$;

(2) $x \neq \varepsilon$;

(3) For all $\alpha, \beta \in \Sigma^*$,

$\alpha u \beta \in L$ iff $\alpha u x^i \beta \in L$ for all $i \geq 0$.

Prove that this pumping lemma holds.

Hint. Use the version of the Myhill-Nerode theorem involving congruences.

(4) Prove that the converse of the pumping lemma in (3) also holds, i.e., if a language $L$ satisfies the pumping lemma in (3), then it is regular.

Hint. Prove that $\rho_L$ has a finite number of equivalence classes.
Problem B2 (60 pts). Give context-free grammars for the following languages:

(a) \( L_5 = \{ wcw^R \mid w \in \{a, b\}^* \} \) (\( w^R \) denotes the reversal of \( w \))

(b) \( L_6 = \{ a^m b^n \mid 1 \leq m \leq n \leq 2m \} \)

For any fixed integer \( K \geq 2 \),
\( L_7 = \{ a^m b^n \mid 1 \leq m \leq n \leq Km \} \)

(c) \( L_8 = \{ a^n b^n \mid n \geq 1 \} \cup \{ a^n b^{2n} \mid n \geq 1 \} \)

(d) \( L_9 = \{ a^m b^n a^m b^p \mid m, n, p \geq 1 \} \cup \{ a^m b^{4n} a^p b^{4n} \mid m, n, p \geq 1 \} \)

(e) \( L_{10} = \{ xcy \mid |x| = 2|y|, x, y \in \{a, b\}^* \} \)

In each case, give a brief justification of the fact that your grammar generates the desired language.

TOTAL: 160 points.