CIS 500 — Software Foundations

Midterm II

Answer key

November 13, 2002
Simply typed lambda-calculus

The definition of the simply typed lambda-calculus with Unit is reproduced on page 11.

1. (8 points) For each of the following untyped λ-terms, either give a well-typed term of the simply typed lambda-calculus with Unit whose erasure is the given term, or else write “not typable” if no such term exists.

The type annotations in your answers should only involve Unit and →.

(a) \(\lambda x. x\ (x\ \text{unit})\)

\[\text{Answer: } \lambda x: \text{Unit} \rightarrow \text{Unit}. x \ (x\ \text{unit})\]

(b) \(\lambda x. x\ \text{unit}\ x\)

\[\text{Answer: Not typable}\]

(c) \(\lambda x. x\ \text{unit}\ \text{unit}\)

\[\text{Answer: } \lambda x: \text{Unit} \rightarrow \text{Unit} \rightarrow \text{Unit}. x\ \text{unit}\ \text{unit}\]

(for example)

(d) \(\lambda x. \lambda y. \lambda z. (x\ y)\ (y\ z)\)

\[\text{Answer: } \lambda x: (\text{Unit} \rightarrow \text{Unit}) \rightarrow \text{Unit} \rightarrow \text{Unit}. \lambda y: \text{Unit} \rightarrow \text{Unit} \rightarrow \text{Unit}. \lambda z: \text{Unit}. (x\ y)\ (y\ z)\]

(for example)

Grading scheme: Each of the items is worth 2 points. Partial credit (1 point) was given for incorrect but “not completely unreasonable” answers.
References

The definition of the simply typed lambda-calculus with references is reproduced on page 11.

2. (9 points) Suppose, for this question, that our language also has let expressions and numbers. Then evaluating the expression

\[
\text{let } x = \text{ref 5 in} \\
\text{let } y = x \text{ in} \\
\text{let } z = \text{ref } (\lambda a: \text{Nat. } y := a; \text{ succ } (!y)) \text{ in} \\
(\!z) (\!y)
\]

beginning in an empty store yields:

Result: 6

Store: 
\[
l_1 \rightarrow 5 \\
l_2 \rightarrow \lambda a: \text{Nat. } l_1 := a; \text{ succ } (!l_1)
\]

Fill in the results and final stores (when started with an empty store) of the following terms:

(a) \[
\text{let } x = \text{ref 2 in} \\
\text{let } y = x \text{ in} \\
\text{let } f = \lambda a: \text{Ref Nat. } \lambda b: \text{Ref Nat. } a := 5; b := 6; !a \text{ in} \\
f \times y
\]

Answer: Result: 6

Store: 
\[
l_1 \rightarrow 6
\]

(b) \[
\text{let } x = \text{ref 2 in} \\
\text{let } y = \text{ref } x \text{ in} \\
\text{let } z = \text{ref } y \text{ in} \\
(z)
\]

Answer: Result: \(l_2\)

Store: 
\[
l_1 \rightarrow 2 \\
l_2 \rightarrow l_1 \\
l_3 \rightarrow l_2
\]

(c) \[
\text{let } x = \text{ref 0 in} \\
\text{let } f = \text{ref } (\lambda u: \text{Unit. } !x) \text{ in} \\
x := 2; \\
\text{let } g = \lambda u: \text{Unit. } (\!f) \text{ unit } \text{ in} \\
x := 3; \\
f := \lambda u: \text{Unit. } \text{ succ } (!x) \text{ in} \\
\text{let } r = g \text{ unit } \text{ in} \\
x := 9; \\
r
\]

Answer: Result: 4

Store: 
\[
l_1 \rightarrow 9 \\
l_2 \rightarrow \lambda u: \text{Unit. } \text{ succ } (!l_1)
\]

Grading scheme: Each of the items was worth 3 points (1 point for the correct result, and 2 points for the correct store). Partial credit was given for "somewhat correct" stores; for instance,

\[
l_2 \rightarrow \lambda u: \text{Unit. } \text{ succ } (!x)
\]

was accepted as an almost correct variant of

\[
l_2 \rightarrow \lambda u: \text{Unit. } \text{ succ } (!l_1).
\]
3. (3 points) Is there any well-typed term that, when started with an empty store, will yield the following store?

\[ l_1 \rightarrow l_1 \]

If so, give one. If not, explain (briefly!) why not.

*Answer: No: this store is not typable, so the preservation theorem tells us that no well-typed program could create it.*

*Grading scheme:*

- 1 point for “no” (0 points for “yes”)
- 2 points for saying something about why it seems hard to create, but omitting the observation about typing.
4. (8 points) We saw in homework 8 that, using references, we can achieve the effect of a recursive function definition by building a “cyclic store” in which the function’s body refers to its own definition indirectly, via a reference cell. The same idea extends straightforwardly to mutually recursive definitions.

Fill in the blanks in the following expressions so that, after evaluating them, even will be a function that checks whether its argument n is even (by returning true if it is 0 and otherwise checking whether (pred n) is odd).

\[
\begin{align*}
even_{\text{ref}} &= \text{ref} (\lambda n: \text{Nat}. \text{true}); \\
\text{odd}_{\text{ref}} &= \text{ref} (\lambda n: \text{Nat}. \text{true}); \\
\text{even}_{\text{body}} &= \lambda n: \text{Nat}. \text{if iszero } n \text{ then true else } (((_______)(\text{pred } n)); \\
\text{odd}_{\text{body}} &= \lambda n: \text{Nat}. \text{if iszero } n \text{ then false else } (((_______)(\text{pred } n)); \\
\text{even}_{\text{ref}} &= \text{______________}; \\
\text{odd}_{\text{ref}} &= \text{______________}; \\
\text{even} &= \text{!even}_{\text{ref}}; \\
\text{odd} &= \text{!odd}_{\text{ref}};
\end{align*}
\]

Answer:

\[
\begin{align*}
\text{even}_{\text{body}} &= \lambda n: \text{Nat}. \text{if iszero } n \text{ then true else } (((\text{!odd}_{\text{ref}})(\text{pred } n)); \\
\text{odd}_{\text{body}} &= \lambda n: \text{Nat}. \text{if iszero } n \text{ then false else } (((\text{!even}_{\text{ref}})(\text{pred } n)); \\
\text{even}_{\text{ref}} &= \text{even}_{\text{body}}; \\
\text{odd}_{\text{ref}} &= \text{odd}_{\text{body}};
\end{align*}
\]

Grading scheme: The problem asked to simulate recursion through references. However, some people defined even$_{\text{body}}$ using odd$_{\text{body}}$, i.e., through the regular recursion. If this did not contain additional errors, 2 points; zero otherwise.

If the dereference operator \! was missing in the entries for even$_{\text{body}}$ and odd$_{\text{body}}$, then minus 4 points.

If the extraneous ref operator was present in the assignments for even$_{\text{ref}}$ and odd$_{\text{ref}}$, then minus 4 points.

If the suffixes ref or body were absent in the function names, minus 2 points.

Minus 2 points for an error in the function logic, e.g. odd instead of even, or an extra negation of a function call, etc.
5. (20 points) In Chapter 13 of TAPL, the following lemmas were used in proving the preservation property for the simply typed lambda-calculus with references. (We've given all the lemmas names here, for easy reference.)

**Lemma [Inversion]:**

(a) If \( \Gamma \vdash x : T \), then \( x : T \in \Gamma \).

(b) If \( \Gamma \vdash \lambda x : T_1 \cdot t_2 : T \), then there is some type \( T_{11} \) such that \( \Gamma \vdash t_1 : T_{11} \rightarrow T \) and \( \Gamma \vdash t_2 : T_{11} \).

(c) If \( \Gamma \vdash t_1 \cdot t_2 : T \), then there is some type \( T_{11} \) such that \( \Gamma \vdash t_1 : T_{11} \rightarrow T \) and \( \Gamma \vdash t_2 : T_{11} \).

(d) If \( \Gamma \vdash \text{unit} : T \), then \( T = \text{Unit} \).

(e) If \( \Gamma \vdash \text{ref } t_1 : T \), then \( T = \text{Ref } T_1 \) and \( \Gamma \vdash t_1 \in T_1 \).

(f) If \( \Gamma \vdash ! t_1 : T \), then \( T = T_{11} \) with \( \Gamma \vdash t_1 \in \text{Ref } T_{11} \).

(g) If \( \Gamma \vdash t_1 := t_2 : T \), then there is some type \( T_{11} \) such that \( \Gamma \vdash t_1 : T_{11} \rightarrow T \) and \( \Gamma \vdash t_2 : T_{11} \).

(h) If \( \Gamma \vdash l : T \), then \( T = \text{Ref } \Sigma (l) \).

**Lemma [Substitution]:** If \( \Gamma, x : S \vdash t : T \) and \( \Gamma \vdash s : S \), then \( \Gamma \vdash [x \mapsto s] t : T \).

**Lemma [Replacement]:** If

\[
\begin{align*}
\Gamma & \vdash \mu \\
\Sigma (l) & = T \\
\Gamma & \vdash v : T
\end{align*}
\]

then \( \Gamma \vdash [l \mapsto v] \mu \).

**Lemma [Weakening]:** If \( \Gamma \vdash t : T \) and \( \Sigma' \supseteq \Sigma \), then \( \Gamma \vdash t : T \).

For each case in the proof on the next page, write down the skeleton of the argument. A skeleton contains the same sequence of steps as the full argument, but omits all details. The rules for writing skeletons are as follows:

- Steps of the form “By part (x) of the inversion lemma, we obtain...” in the full argument become “inversion(x)” in the skeleton.
- Steps of the form “By the substitution lemma, we obtain...” become “substitution.” (Similarly for replacement and weakening.)
- Steps of the form “By the induction hypothesis, we obtain...” become “IH.”
- Steps of the form “By typing rule T-XXX, we obtain...” become “T-XXX.”
- If the full argument doesn’t use any of the lemmas or the induction hypothesis, then its skeleton is “Direct.”

For example, if the full argument is

**Case E-DEREFLOC:** \( t = !l \quad t' = \mu (l) \quad \mu' = \mu \)

By part (f) of the inversion lemma, \( T = T_{11} \), and \( \Gamma \vdash l : \text{Ref } T_{11} \). By part (h) of the inversion lemma, \( T_{11} = \text{Ref } \Sigma (l) \), i.e., \( T = T_{11} = \Sigma (l) \). But now, from the assumption that \( \Gamma \vdash \mu \), we can conclude (by the definition of \( \Gamma \vdash \mu \)) that \( \Gamma \vdash \mu (l) : \Sigma (l) \).

the skeleton is written:

**Case E-DEREFLOC:** \( t = !l \quad t' = \mu (l) \quad \mu' = \mu \)

Inversion(f), inversion(h)

As a second example, the case for E-REF is also given below.
**Theorem [Preservation]:** If

\[ \Gamma \vdash \Sigma \vdash t : T \]
\[ \Gamma \vdash \Sigma \vdash \mu \quad \text{(i.e.,} \quad \text{dom}(\mu) = \text{dom}(\Sigma) \text{ and} \quad \Gamma \vdash \mu(l) : \Sigma(l) \text{ for every} \ l \in \text{dom}(\mu)) \]
\[ t \mid \mu \rightarrow t' \mid \mu' \]

then, for some \( \Sigma' \supseteq \Sigma \),
\[ \Gamma \vdash \Sigma' \vdash t' : T \]
\[ \Gamma \vdash \Sigma' \vdash \mu'. \]

**Proof:** By induction on evaluation derivations, with a case analysis on the final rule used.

**Case E-App1:** \( t = t_1 \ t_2 \quad t_1 \mid \mu \rightarrow t'_1 \mid \mu' \quad t' = t'_1 \ t_2 \)

Answer: Inversion(c), IH, weakening, T-App.

**Case E-App2:**

Similar.

**Case E-AppAbs:** \( t = (\lambda x: T_1 \rightarrow T_2) \ v_2 \quad t' = [x \mapsto v_2]_{T_1} \quad \mu' = \mu \)

Answer: inversion(c), inversion(b), substitution

**Case E-Ref:** \( t = \text{ref} \ t_1 \quad t' = \text{ref} \ t'_1 \quad t_1 \mid \mu \rightarrow t'_1 \mid \mu' \)

inversion(e), IH, T-Ref

**Case E-DerefLoc:** \( t = !l \quad t' = \mu(l) \quad \mu' = \mu \)

Inversion(f), inversion(h)

**Case E-Deref:** \( !t_1 \mid \mu \rightarrow !t'_1 \mid \mu' \)

Answer: inversion(f), IH, T-Deref

**Case E-Assign:** \( t = l := v_2 \quad t' = \text{unit} \quad \mu' = [l \mapsto v_2]_{\mu} \)

Answer: inversion(g), inversion(h), replacement, T-Unit

**Case E-Assign1:** \( t = t_1 := t_2 \quad t' = t'_1 := t_2 \quad t_1 \mid \mu \rightarrow t'_1 \mid \mu' \)

Answer: inversion(g), IH, weakening, T-Assign

**Case E-Assign2:**

Similar.

**Grading scheme:** 4 points possible for each part

- 1 point off for missing weakening, inversion, or use of a typing rule
- 2 points off for missing IH, substitution, or replacement
- 1 point off for wrong ordering
- 1 point off for using the wrong case of the inversion lemma
- 1 point off for including an unnecessary step, as long the extra step was possible; 2 points if the extra step was actually wrong (not legal at the point where it appeared)
- 3 points off whole problem for writing a full proof instead of a sketch
Subtyping

The definition of the simply typed lambda-calculus with records and subtyping is reproduced for your reference on page 13.

6. (11 points) For each type $S$ from the left-hand column below, draw a line connecting it to each type $T$ in the right-hand column such that $S <: T$.

<table>
<thead>
<tr>
<th>Choices for $S$:</th>
<th>Choices for $T$:</th>
</tr>
</thead>
<tbody>
<tr>
<td>${a:{}$, $b:{x:Top}}$</td>
<td>${} \rightarrow {a:Top} \rightarrow {}$</td>
</tr>
<tr>
<td>Top $\rightarrow$ Top</td>
<td>Top $\rightarrow$ Top</td>
</tr>
<tr>
<td>${} \rightarrow {}$</td>
<td>${} \rightarrow$ Top</td>
</tr>
<tr>
<td>Top</td>
<td>Top $\rightarrow {}$</td>
</tr>
<tr>
<td>$({a:Top} \rightarrow {}) \rightarrow {b:Top}$</td>
<td>${b:Top}$</td>
</tr>
<tr>
<td>${b:Top \rightarrow$ Top$}$</td>
<td>${b:{}}$</td>
</tr>
</tbody>
</table>

Answer: Numbering both columns from top to bottom, we have

- $S_1 <: T_5$, $T_6$
- $S_2 <: T_2$, $T_3$
- $S_3 <: T_3$
- $S_4$ is not a subtype of any of the $T$s
- $S_5 <: T_1$
- $S_6 <: T_5$

Grading scheme: 1 point off for each missing line; 1 off for each incorrect line.
7. (12 points) It is easy to show, by induction on subtyping derivations, that

**Lemma A:** If $\text{Top} <: T$, then $T = \text{Top}$.

A similar, but slightly more interesting, lemma holds for supertypes of arrow types.

**Lemma B:** If $S_1 \rightarrow S_2 <: T$, then either $T = \text{Top}$ or else $T$ has the form $T_1 \rightarrow T_2$, with $T_1 <: S_1$ and $S_2 <: T_2$.

Fill in the arguments for the S-Arrow and S-Trans cases of its proof.

**Proof:** By induction on subtyping derivations. Proceed by a case analysis on the last rule used in the derivation.

**Case S-Refl:** $T = S_1 \rightarrow S_2$

$T$ clearly has the required form, with $T_1 = S_1$ and $T_2 = S_2$. The inclusions $T_1 <: S_1$ and $S_2 <: T_2$ both follow by S-Refl.

**Case S-Trans:** $S_1 \rightarrow S_2 <: U$, $U <: T$

**Answer:**

By the induction hypothesis, either $U = \text{Top}$ or else $U$ has the form $U_1 \rightarrow U_2$, with $U_1 <: S_1$ and $S_2 <: U_2$. In the first case ($U = \text{Top}$), lemma A tells us that $T = \text{Top}$ and we are finished. Otherwise, $U = U_1 \rightarrow U_2$, and we can use the induction hypothesis again to show that either $T = \text{Top}$ or else $T$ has the form $T_1 \rightarrow T_2$, with $T_1 <: U_1$ and $U_2 <: T_2$. But now, from $T_1 <: U_1$ and $U_1 <: S_1$, we obtain $T_1 <: S_1$ using S-Trans. Similarly, from $S_2 <: U_2$ and $U_2 <: T_2$, S-Trans yields $S_2 <: T_2$.

**Case S-Arrow:** $T = T_1 \rightarrow T_2$, $T_1 <: S_1$ and $S_2 <: T_2$

**Answer:**

Immediate.

**Case S-Top:** $T = \text{Top}$

Immediate.

**Case S-RcdWidth, S-RcdDepth, S-RcdPerm, S-Top:**

Can’t happen: $T$ has the wrong form.

**Grading scheme:** The subcase S-Trans was worth 9 points; subcase S-Arrow, 3 points.

Minus 1 point for each case when, in an otherwise correct proof, a step was not justified by an explicit reference to an inductive case, Lemma A, or the rule S-Trans. Minus 1 point when, instead of IH, the proof referred to Lemma B.

Minus 3 points when the consideration of the Top subcase in the induction step is omitted completely.

At most 4 points (out of 9) were given when the induction argument was very unclear but most of the statements that would arise in a correct proof were mentioned.

Some solutions got 0 points for totally incoherent arguments.
8. (9 points) Suppose we remove rule S-ARROW from the subtype relation. Which of the following properties will remain true? For each one, write either “true” (if it remains true) or else “false” (if it becomes false), plus (in either case) a one-sentence justification of your answer.

(a) Existence of minimal types (if term \( t \) is typable in context \( \Gamma \), then there is some type \( S \) such that \( \Gamma \vdash t : S \) and, for every type \( T \) such that \( \Gamma \vdash t : T \), we have \( S <: T \))
Answer: False. For example, the term \( \lambda x:{}. x \) has both the types \( {} \rightarrow {} \) and \( {} \rightarrow \text{Top} \), but, without the arrow rule, these two types have no common lower bound.

(b) Progress (if \( t \) is a closed, well-typed term, then either \( t \) is a value or else \( t \rightarrow t' \) for some \( t' \))
Answer: True: removing pairs from the subtype relation can only reduce the number of well-typed terms, which can only make it easier for progress to hold.

(c) Preservation (if \( t \) has type \( T \) and \( t \rightarrow t' \), then \( t' \) also has type \( T \))
Answer: True. The only part of the preservation proof that changes is the inversion lemma (which changes back to its form from chapter 9(!) and becomes easier to prove) and the case of the main proof where it is used.

This part of the question was more subtle than the others, since it is not the case that preservation remains true for any simple reason. In particular, it's not correct to observe, at this point, that making fewer terms well typed can only make preservation easier by weakening its premise, since its conclusion is also weakened. Indeed, it could very well be that removing things from the subtype relation could break preservation; it just happens that, in this case, it does not.

Grading scheme: -3 for each wrong answer. One point awarded for correct answer. One point awarded for partial explanation (given generously). One point awarded for complete explanation (given sparingly).
For reference: Simply typed lambda calculus with `Unit`

**Syntax**

\[ t ::= \]

- `unit`
- `x`
- `\lambda x : T . t`
- `t t`

\[ v ::= \]

- `unit`
- `\lambda x : T . t`

\[ T ::= \]

- `Unit`
- `T \rightarrow T`

\[ \Gamma ::= \]

- `\emptyset`
- `\Gamma, x : T`

**Evaluation**

\[ t_1 \rightarrow t'_1 \]

\[ t_1 t_2 \rightarrow t'_1 t_2 \]

\[ t_2 \rightarrow t'_2 \]

\[ v_1 t_2 \rightarrow v'_1 t'_2 \]

\[(\lambda x : T_{11} . t_{12}) v_2 \rightarrow [x \mapsto v_2] t_{12}\]

**Typing**

\[ \Gamma \vdash t : T \]

\[ x : T \in \Gamma \]

\[ \Gamma \vdash x : T \]

\[ \Gamma, x : T_1 \vdash t_2 : T_2 \]

\[ \Gamma \vdash \lambda x : T_1 . t_2 : T_1 \rightarrow T_2 \]

\[ \Gamma \vdash t_1 : T_{11} \rightarrow T_{12} \]

\[ \Gamma \vdash t_2 : T_{11} \]

\[ \Gamma \vdash t_1 \ t_2 : T_{12} \]
For reference: References

New syntactic forms
\[ t ::= \ldots \]
\[ \text{reference creation} \]
\[ !t \]
\[ t:=t \]
\[ l \]
\[ \text{dereference} \]
\[ v ::= \ldots \]
\[ l \]
\[ \text{assignment} \]
\[ T ::= \ldots \]
\[ \text{type of reference cells} \]
\[ \text{store location} \]

New evaluation rules
\[ t \mid \mu \rightarrow t' \mid \mu' \]

\[ (\lambda x : T_1. t_2) v_2 \mid \mu \rightarrow [x \rightarrow v_2] t_2 \mid \mu \]
\[ (E-APPABS) \]

\[ l \notin \text{dom}(\mu) \]
\[ \text{empty store} \]
\[ \text{location binding} \]
\[ \text{store typings} \]
\[ \text{empty store typing} \]
\[ \text{location typing} \]

\[ \begin{align*} 
\tau_1 \mid \mu & \rightarrow \tau'_1 \mid \mu' \\
\tau_1 \tau_2 \mid \mu & \rightarrow \tau'_1 \tau'_2 \mid \mu' \\
v_1 \tau_2 \mid \mu & \rightarrow v_1 \tau'_2 \mid \mu' \\

\end{align*} \]
\[ (E-APP1) \]
\[ (E-APP2) \]
\[ (E-RefV) \]
\[ (E-DerefLoc) \]
\[ (E-Deref) \]
\[ (E-Assign) \]
\[ (E-Assign1) \]
\[ (E-Assign2) \]
New typing rules

\[
\begin{align*}
\Gamma | \Sigma \vdash \text{unit} : \text{Unit} & \quad \text{(T-UNIT)} \\
\Gamma | \Sigma \vdash x : T & \quad \text{(T-VAR)} \\
\Gamma, x : T_1 | \Sigma \vdash t_2 : T_2 & \quad \text{(T-ABS)} \\
\Gamma \vdash \lambda x : T_1 . t_2 : T_1 \rightarrow T_2 & \quad \text{(T-APP)} \\
\Gamma | \Sigma \vdash t_1 : T_{11} \rightarrow T_{12} & \quad \text{(T-LOC)} \\
\Gamma | \Sigma \vdash t_1 : T_{12} & \quad \text{(T-LOC)} \\
\Gamma | \Sigma \vdash t_1 : T_{12} & \quad \text{(T-LOC)} \\
\Gamma | \Sigma \vdash \text{ref } t_1 : \text{Ref } T_1 & \quad \text{(T-LOC)} \\
\Gamma | \Sigma \vdash t_1 : \text{Ref } T_{11} & \quad \text{(T-LOC)} \\
\Gamma | \Sigma \vdash \text{!} t_1 : T_{11} & \quad \text{(T-LOC)} \\
\Gamma | \Sigma \vdash t_1 : \text{Ref } T_{11} & \quad \text{(T-LOC)} \\
\Gamma | \Sigma \vdash t_1 : \text{Ref } T_{11} & \quad \text{(T-LOC)} \\
\Gamma | \Sigma \vdash \text{ref } t_1 : \text{Ref } T_1 & \quad \text{(T-LOC)} \\
\Gamma | \Sigma \vdash t_1 : \text{Ref } T_{11} & \quad \text{(T-LOC)} \\
\Gamma | \Sigma \vdash \text{!} t_1 : T_{11} & \quad \text{(T-LOC)} \\
\Gamma | \Sigma \vdash t_1 := t_2 : \text{Unit} & \quad \text{(T-ASSIGN)} \\
\end{align*}
\]
For reference: Simply typed lambda calculus with records and subtyping

**New syntactic forms**

\[
\begin{align*}
\text{t} & ::= \ldots \{l_i=t_i \mid i \in 1..n\} \text{ terms} \\
\text{v} & ::= \ldots \{l_i=v_i \mid i \in 1..n\} \text{ values} \\
\text{T} & ::= \ldots \{l_i:T_i \mid i \in 1..n\} \text{ types}
\end{align*}
\]

**New evaluation rules**

\[
\begin{align*}
&\{l_i=v_i \mid i \in 1..n\} \cdot l_j \rightarrow v_j \\
&t_1 \rightarrow t'_1 \\
&t_1 \cdot l \rightarrow t'_1 \cdot l \\
&t_j \rightarrow t'_j \\
&\{l_i=v_i \mid i \in 1..j-1\}, l_j=t_j, l_k=t_k \mid k \in j+1..n\} \rightarrow \{l_i=v_i \mid i \in 1..j-1\}, l_j=t'_j, l_k=t'_k \mid k \in j+1..n\}
\end{align*}
\]

**New subtyping rules**

\[
\begin{align*}
S &<: S \quad \text{(S-REFL)} \\
S &<: U \quad U <: T \quad \text{(S-TRANS)} \\
S &<: T \quad \text{(S-TOP)} \\
T_1 &<: S_1 \quad S_2 <: T_2 \quad T_1 - S_2 &<: T_1 - T_2 \quad \text{(S-ARROW)} \\
\{l_i:T_i \mid i \in 1..n\} &<: \{l_i:T_i \mid i \in 1..n\} \quad \text{(S-RCDWIDTH)} \\
\text{for each } i \quad S_i &<: T_i \quad \{l_i:S_i \mid i \in 1..n\} &<: \{l_i:T_i \mid i \in 1..n\} \quad \text{(S-RCDDEPTH)} \\
\{k_j:S_j \mid j \in 1..n\} \text{ is a permutation of } \{l_i:T_i \mid i \in 1..n\} \quad \{k_j:S_j \mid j \in 1..n\} &<: \{l_i:T_i \mid i \in 1..n\} \quad \text{(S-RCDPERM)}
\end{align*}
\]

**New typing rules**

\[
\begin{align*}
&\text{for each } i \quad \Gamma \vdash t_i : T_i \\
&\Gamma \vdash \{l_i=t_i \mid i \in 1..n\} : \{l_i:T_i \mid i \in 1..n\} \\
&\Gamma \vdash t_1 : \{l_i:T_i \mid i \in 1..n\} \\
&\Gamma \vdash t_1 \cdot l_j : T_j \\
&\Gamma \vdash t : S \quad S <: T \\
&\Gamma \vdash t : T \quad \text{(T-SUB)}
\end{align*}
\]