

Metatheory of Typing

For the typing relation, we have just one problematic rule to deal with:
subsumption.

$$\frac{\Gamma \vdash t : T}{\Gamma \vdash t : S} \text{ S} \leq \text{ T}$$

(T-Sub)

We observed last time that this rule is sometimes **required** when typechecking applications:
E.g., the term

```
( $\lambda r : \{x : \text{Nat}\}. r \cdot x$ ). {x=0,y=1}
```

is not typable without using subsumption.
But we conjectured that applications were the only critical uses of subsumption.

CIS 500
 Software Foundations
 Fall 2004
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Midterm 2

- ◆ Midterm answers are on the website
- ◆ Regrade procedure: prepare a **specific written description** of grading errors and submit with your exam to Cheryl Hickey. Deadline, Dec. 8th
- ◆ Statistics on Exam
 - ◆ Minimum: 19
 - ◆ Maximum: 79
 - ◆ Median : 48

Plan

1. Investigate how subsumption is used in typing derivations by looking at examples of how it can be “pushed through” other rules
2. Use the intuitions gained from this exercise to design a new, algorithmic typing relation that
 - ◆ omits subsumption
 - ◆ compensates for its absence by enriching the application rule
3. Show that the algorithmic typing relation is essentially equivalent to the original, declarative one

Example (T-SUB with T-ABS)

$$\begin{array}{c}
 \vdots \\
 \frac{\Gamma, x:s_1 \vdash s_2 : T_2}{\Gamma, x:s_1 \vdash s_2 : T_2} \text{(T-SUB)} \\
 \frac{\Gamma, x:s_1 \vdash s_2 : T_2}{\Gamma \vdash \lambda x:s_1.s_2 : s_1 \rightarrow T_2} \text{(T-ABS)}
 \end{array}$$

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 \end{array}$$

becomes

$$\begin{array}{c}
 \vdots \\
 \frac{\Gamma, x:s_1 \vdash s_2 : T_2}{\Gamma, x:s_1 \vdash s_2 : T_2} \text{(T-ABS)} \\
 \frac{\Gamma, x:s_1 \vdash s_2 : T_2}{\Gamma, x:s_1 \vdash s_2 : T_2} \text{(S-REFL)} \\
 \frac{s_1 <: s_1}{s_1 <: T_2} \text{(S-ARROW)} \\
 \frac{s_1 \rightarrow s_2 <: s_1 \rightarrow T_2}{\Gamma \vdash \lambda x:s_1.s_2 : s_1 \rightarrow T_2} \text{(T-SUB)}
 \end{array}$$

[board]

Example (T-SUB with T-RCD)

Summary

What we've learned:

- ◆ Uses of the T-SUB rule can be “pushed down” through typing derivations until they encounter either
 1. a use of T-APP or
 2. the root of the derivation tree.
 - ◆ In both cases, multiple uses of T-SUB can be collapsed into a single one.
- This suggests a notion of “normal form” for typing derivations, in which there is
- ◆ exactly one use of T-SUB before each use of T-APP
 - ◆ one use of T-SUB at the very end of the derivation
 - ◆ no uses of T-SUB anywhere else.

Summary

What we've learned:

- ◆ Uses of the T-SUB rule can be “pushed down” through typing derivations until they encounter either
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 2. the root of the derivation tree.
- ◆ In both cases, multiple uses of T-SUB can be collapsed into a single one.

Algorithmic Typing

The next step is to “build in” the use of substitution in application rules, by changing the T-APP rule to incorporate a substituting premise.

$$\frac{\Gamma \vdash t_1 \quad t_2 : \tau_2 \quad \Gamma \vdash t_1 \rightarrow t_2 \quad \Gamma \vdash t_2 : \tau_2 \quad \Gamma \vdash t_1 \rightarrow t_2 : \tau_2}{\Gamma \vdash t_1 \quad t_2 : \tau_2}$$

Given any typing derivation, we can now

1. normalize it, to move all uses of substitution to either just before applications (in the right-hand premise) or at the very end
 2. replace uses of T-APP with T-SUB in the right-hand premise by uses of the extended rule above
- This yields a derivation in which there is just **one** use of substitution, at the very end!

Minimal Types

But... if substitution is only used at the very end of derivations, then it is actually **not needed** in order to show that any term is typable!

It is just used to give **more** types to terms that have already been shown to have a type.

In other words, if we dropped substitution completely (after refining the application rule), we would still be able to give types to exactly the same set of terms — we just would not be able to give as many types to some of them.

If we drop substitution, then the remaining rules will assign a **unique, minimal** type to each typable term.

For purposes of building a typechecking algorithm, this is enough.

Final Algorithmic Typing Rules

$$\text{(TA-VAR)} \quad \frac{x : T \in \Gamma}{\Gamma \vdash x : T}$$

$$\text{(TA-ABS)} \quad \frac{\Gamma, x : T_1 \vdash t_2 : T_2}{\Gamma \vdash \lambda x : T_1. t_2 : T_1 \rightarrow T_2}$$

$$\text{(TA-APP)} \quad \frac{\Gamma \vdash t_1 : T_1 \quad \Gamma \vdash t_2 : T_2 \quad \Gamma \vdash t_1 \rightarrow T_2 \rightarrow T_1}{\Gamma \vdash t_1 t_2 : T_1 T_2}$$

$$\text{(TA-RCD)} \quad \frac{\text{for each } i \quad \Gamma \vdash t_i : T_i}{\Gamma \vdash \{l_1 = t_1 \dots l_n = t_n\} : \{l_1 : T_1 \dots l_n : T_n\}}$$

$$\text{(TA-PROJ)} \quad \frac{\Gamma \vdash t_1, l_1 : T_1}{\Gamma \vdash t_1 : R_1 \quad R_1 = \{l_1 : T_1 \dots l_n : T_n\}}$$

Completeness of the algorithmic rules

Theorem [Minimal Typing]: If $\Gamma \vdash t : T$, then $\Gamma \vdash t : S$ for some $S < T$.

Soundness of the algorithmic rules

Theorem: If $\Gamma \vdash t : T$, then $\Gamma \vdash t : T$.

(N.b.: All the messing around with transforming derivations was just to build intuitions and decide what algorithmic rules to write down and what property to prove: the proof itself is a straightforward induction on typing derivations.)

Proof: Homework.

Theorem [Minimal Typing]: If $\Gamma \vdash t : T$, then $\Gamma \vdash t : S$ for some $S < T$.

Completeness of the algorithmic rules

A Problem with Conditional Expressions

For the **algorithmic** presentation of the system, however, we encounter a little difficulty.

What is the minimal type of

`if true then {x=true,y=false} else {x=true,z=true}`?

The Algorithmic Conditional Rule

$$\frac{\Gamma \vdash t_1 \text{ then } t_2 \text{ else } t_3 : T_2 \vee T_3}{\Gamma \vdash t_1 : \text{Bool} \quad \Gamma \vdash t_2 : T_2 \quad \Gamma \vdash t_3 : T_3}$$

More generally, we can use subsumption to give an expression `if t1 then t2 else t3` any type that is a possible type of both `t2` and `t3`. So the **minimal** type of the conditional is the **least common supertype** (or **join**) of the minimal type of `t2` and the minimal type of `t3`.

Meets and Joins

Adding Booleans

Suppose we want to add booleans and conditionals to the language we have been discussing.

For the **declarative** presentation of the system, we just add in the appropriate syntactic forms, evaluation rules, and typing rules.

$$\frac{\Gamma \vdash t_1 \text{ then } t_2 \text{ else } t_3 : T}{\Gamma \vdash t_1 : \text{Bool} \quad \Gamma \vdash t_2 : T \quad \Gamma \vdash t_3 : T}$$

(T-TRUE) $\Gamma \vdash \text{true} : \text{Bool}$

(T-FALSE) $\Gamma \vdash \text{false} : \text{Bool}$

(T-IF) $\Gamma \vdash \text{if } t_1 \text{ then } t_2 \text{ else } t_3 : T$

Examples

- What are the joins of the following pairs of types?
- $\{x:\text{Bool}, y:\text{Bool}\}$ and $\{y:\text{Bool}, z:\text{Bool}\}$?
 - $\{x:\text{Bool}\}$ and $\{y:\text{Bool}\}$?
 - $\{x:\{a:\text{Bool}, b:\text{Bool}\}\}$ and $\{x:\{b:\text{Bool}, c:\text{Bool}\}, y:\text{Bool}\}$?
 - $\{\}$ and Bool ?
 - $\{x:\{\}\}$ and $\{x:\text{Bool}\}$?
 - $\text{Top} \rightarrow \{x:\text{Bool}\}$ and $\text{Top} \rightarrow \{y:\text{Bool}\}$?
 - $\{x:\text{Bool}\} \rightarrow \text{Top}$ and $\{y:\text{Bool}\} \rightarrow \text{Top}$?

Meets

To calculate joins of arrow types, we also need to be able to calculate **meets** (greatest lower bounds)!

Unlike joins, meets do not necessarily exist.

E.g., $\text{Bool} \rightarrow \text{Bool}$ and $\{\}$ have **no** common subtypes, so they certainly don't have a greatest one!

However...

The Algorithmic Conditional Rule

More generally, we can use substitution to give an expression

$$\text{if } t_1 \text{ then } t_2 \text{ else } t_3$$

any type that is a possible type of both t_2 and t_3 .

So the **minimal** type of the conditional is the **least common supertype** (or **join**) of the minimal type of t_2 and the minimal type of t_3 .

$$\frac{\Gamma \vdash t_1 : \text{Bool} \quad \Gamma \vdash t_2 : T_2 \quad \Gamma \vdash t_3 : T_3}{\Gamma \vdash \text{if } t_1 \text{ then } t_2 \text{ else } t_3 : T_2 \vee T_3} \text{ (T-If)}$$

Does such a type exist for every T_2 and T_3 ??

Existence of Joins

Theorem: For every pair of types S and T , there is a type J such that

- $S \leq J$
- $T \leq J$
- If K is a type such that $S \leq K$ and $T \leq K$, then $J \leq K$.

I.e., J is the smallest type that is a supertype of both S and T .

$$S \vee T = \left\{ \begin{array}{ll} \text{Bool} & \text{if } S = T = \text{Bool} \\ M_1 \rightarrow J_2 & \text{if } S = S_1 \rightarrow S_2 \quad T = T_1 \rightarrow T_2 \\ \{j_1: J_1 \mid \ell \in \ell \dots q\} & \text{if } S = \{k_j: S_j \mid \ell \in \ell \dots m\} \\ & T = \{l_i: T_i \mid \ell \in \ell \dots n\} \\ & \{j_1 \mid \ell \in \ell \dots q\} = \{k_j \mid \ell \in \ell \dots m\} \cup \{l_i \mid \ell \in \ell \dots n\} \\ & S_j \vee T_i = J_1 \quad \text{for each } j_1 = k_j = l_i \\ \text{Top} & \text{otherwise} \end{array} \right.$$

Calculating Joins

Theorem: For every pair of types S and T , if there is any type N such that $N < S$ and $N < T$, then there is a type M such that

- $M < S$
- $M < T$
- If O is a type such that $O < S$ and $O < T$, then $O < M$.

I.e., M (when it exists) is the largest type that is a subtype of both S and T .

Jargon: In the simply typed lambda calculus with subtyping, records, and booleans...

- ◆ The subtype relation has **joins**
- ◆ The subtype relation has **bounded meets**

Existence of Meets

$$S \vee T = \left\{ \begin{array}{ll} \text{S} & \text{if } T = \text{Top} \\ \text{T} & \text{if } S = \text{Top} \\ \text{Bool} & \text{if } S = T = \text{Bool} \\ J_1 \rightarrow M_2 & \text{if } S = S_1 \rightarrow S_2 \quad T = T_1 \rightarrow T_2 \\ & S_1 \vee T_1 = J_1 \quad S_2 \vee T_2 = M_2 \\ \{m_1: M_1 \mid \ell \in \ell \dots q\} & \text{if } S = \{k_j: S_j \mid \ell \in \ell \dots m\} \\ & T = \{l_i: T_i \mid \ell \in \ell \dots n\} \\ & \{m_1 \mid \ell \in \ell \dots q\} = \{k_j \mid \ell \in \ell \dots m\} \cup \{l_i \mid \ell \in \ell \dots n\} \\ & \text{for each } m_1 = k_j = l_i \\ S_j \vee T_i = M_1 & \text{for each } m_1 = k_j = l_i \\ M_1 = S_j & \text{if } m_1 = k_j \text{ occurs only in } S \\ M_1 = T_i & \text{if } m_1 = l_i \text{ occurs only in } T \\ \text{fail} & \text{otherwise} \end{array} \right.$$

Calculating Meets

What are the meets of the following pairs of types?

- $\{x:\text{Bool}, y:\text{Bool}\}$ and $\{y:\text{Bool}, z:\text{Bool}\}$?
- $\{x:\text{Bool}\}$ and $\{y:\text{Bool}\}$?
- $\{x:\{a:\text{Bool}, b:\text{Bool}\}\}$ and $\{x:\{b:\text{Bool}, c:\text{Bool}\}, y:\text{Bool}\}$?
- $\{\}$ and Bool ?
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- $\{x:\text{Bool}\} \rightarrow \text{Top}$ and $\{y:\text{Bool}\} \rightarrow \text{Top}$?

Examples