Online Learning: LMS and Perceptrons

Partially adapted from slides by Ryan Gabbard

and Mitch Marcus (and lots original slides by Lyle Ungar)

Learning Objectives

Complexity of OLS

LMS = SDG

Perceptron variations

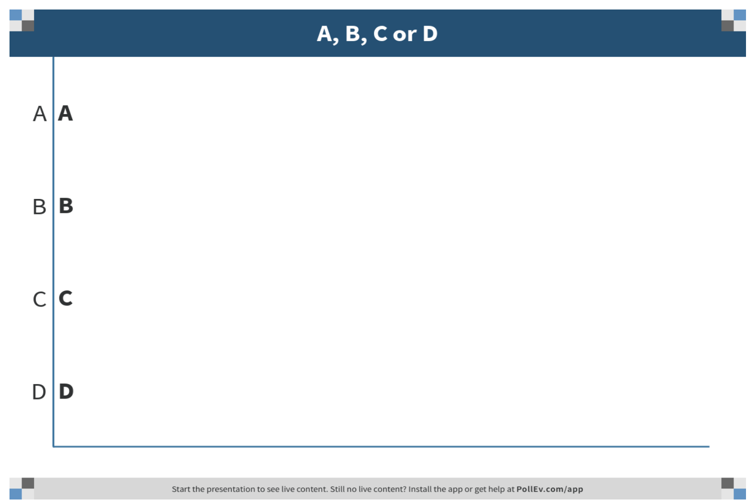
online hinge loss optimization

Note: not on midterm

Why do online learning?

- **Batch learning can be expensive for big datasets**
  - How expensive is it to compute $(X^TX)^{-1}$?

A) n^3
B) p^3
C) np^2
D) n^2p



Why do online learning?

- **Batch learning can be expensive for big datasets**
  - How hard is it to compute $(X^TX)^{-1}$?
    - np^2 to form X^TX
    - p^3 to invert
  - Tricky to parallelize inversion
- **Online methods are easy in a map-reduce environment**
  - They are often clever versions of stochastic gradient descent

Have you seen map-reduce/hadoop?

A) Yes
B) No

Online learning methods

- **Least Mean Squares (LMS)**
  - Online regression -- L_2 error
- **Perceptron**
  - Online SVM -- Hinge loss

LMS: Online linear regression

- **Minimize** Err = $\Sigma_i (y_i - \mathbf{w}^T\mathbf{x}_i)^2$ **using stochastic gradient descent**
  - Look at each observation ($\mathbf{x}_i$, y_i) sequentially and decrease its error $Err_i = (y_i - \mathbf{w}^T\mathbf{x}_i)^2$
- **LMS (Least Mean Squares) algorithm**
  - $\mathbf{w}_{i+1} = \mathbf{w}_i - \eta/2 \, dErr_i/d\mathbf{w}_i$
  - $dErr_i/d\mathbf{w}_i = -2 (y_i - \mathbf{w}_i^T\mathbf{x}_i) \mathbf{x}_i = -2 r_i \mathbf{x}_i$

  $$\mathbf{w}_{i+1} = \mathbf{w}_i + \eta \, r_i \mathbf{x}_i$$

How do you pick the "learning rate" η?

Note that *i* is the index for both the iteration and the observation, since there is one update per observation

Online linear regression

- **LMS (Least Mean Squares) algorithm**

  $$\mathbf{w}_{i+1} = \mathbf{w}_i + \eta\, r_i\, \mathbf{x}_i$$

- **Converges for** $0 < \eta < \lambda_{max}$
  - Where λ_{max} is the largest eigenvalue of the covariance matrix $\mathbf{X}^T\mathbf{X}$

- **Convergence rate is proportional to** $\lambda_{min}/\lambda_{max}$ (ratio of extreme eigenvalues of $\mathbf{X}^T\mathbf{X}$)

Perceptron Learning Algorithm

Input: A list T of training examples $\langle \vec{x}_0, y_0 \rangle \ldots \langle \vec{x}_n, y_n \rangle$ where
$\forall i : y_i \in \{+1, -1\}$

Output: A classifying hyperplane $\vec{w}$

Randomly initialize $\vec{w}$;

while *model* $\vec{w}$ *makes errors on the training data* **do**
  for $\langle \vec{x}_i, y_i \rangle$ *in* T **do**
    Let $\hat{y} = sign(\vec{w} \cdot \vec{x}_i)$;
    if $\hat{y} \neq y_i$ **then**
      $\vec{w} = \vec{w} + y_i \vec{x}_i$;
    end
  end
end

If you were wrong, make *w* look more like *x*

What do we do if error is zero?

Of course, this only converges for linearly separable data

Perceptron Learning Algorithm

For each observation ($\mathbf{x}_i$, y_i)

$$\mathbf{w}_{i+1} = \mathbf{w}_i + \eta \; r_i \, \mathbf{x}_i$$

Where $r_i = y_i - \text{sign}(\mathbf{w}_i^T\mathbf{x}_i)$

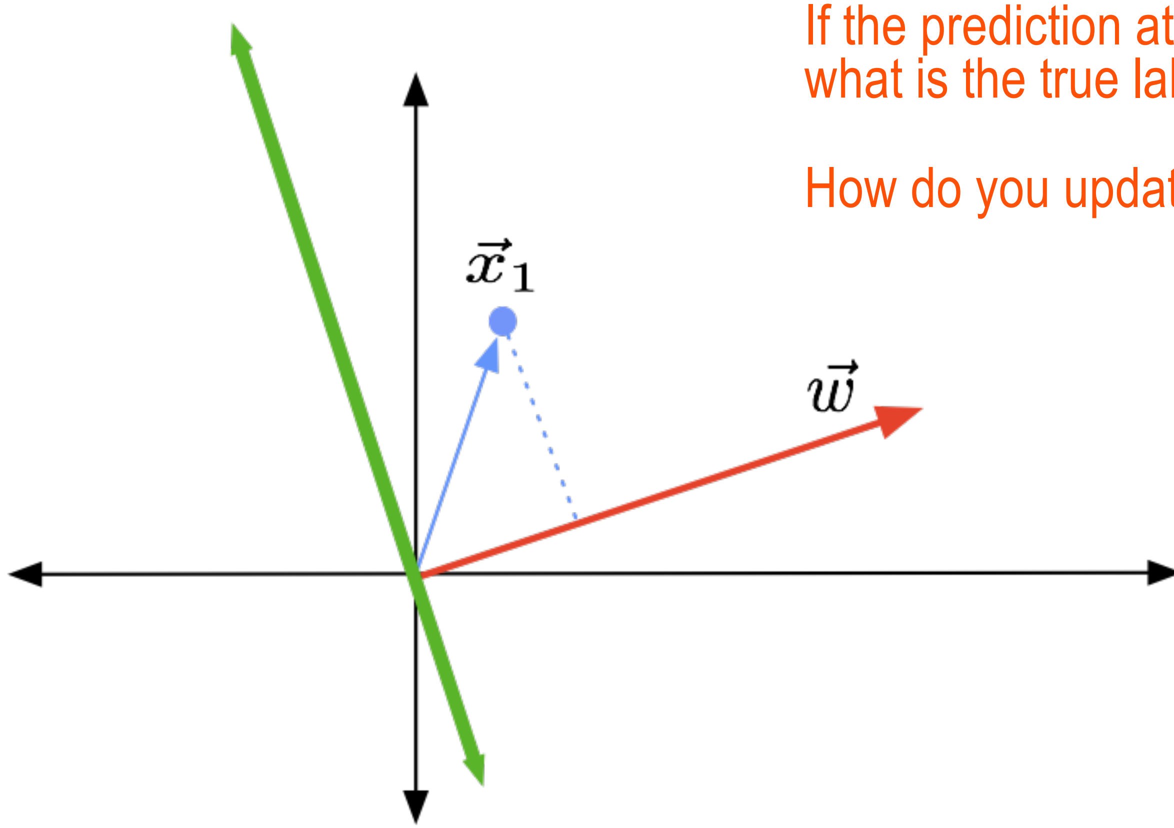
and $\eta = ½$

I.e., if we get it right: *no change*

if we got it wrong: $\mathbf{w}_{i+1} = \mathbf{w}_i + y_i \, \mathbf{x}_i$

Ho does this relate to SVMs?

Perceptron update

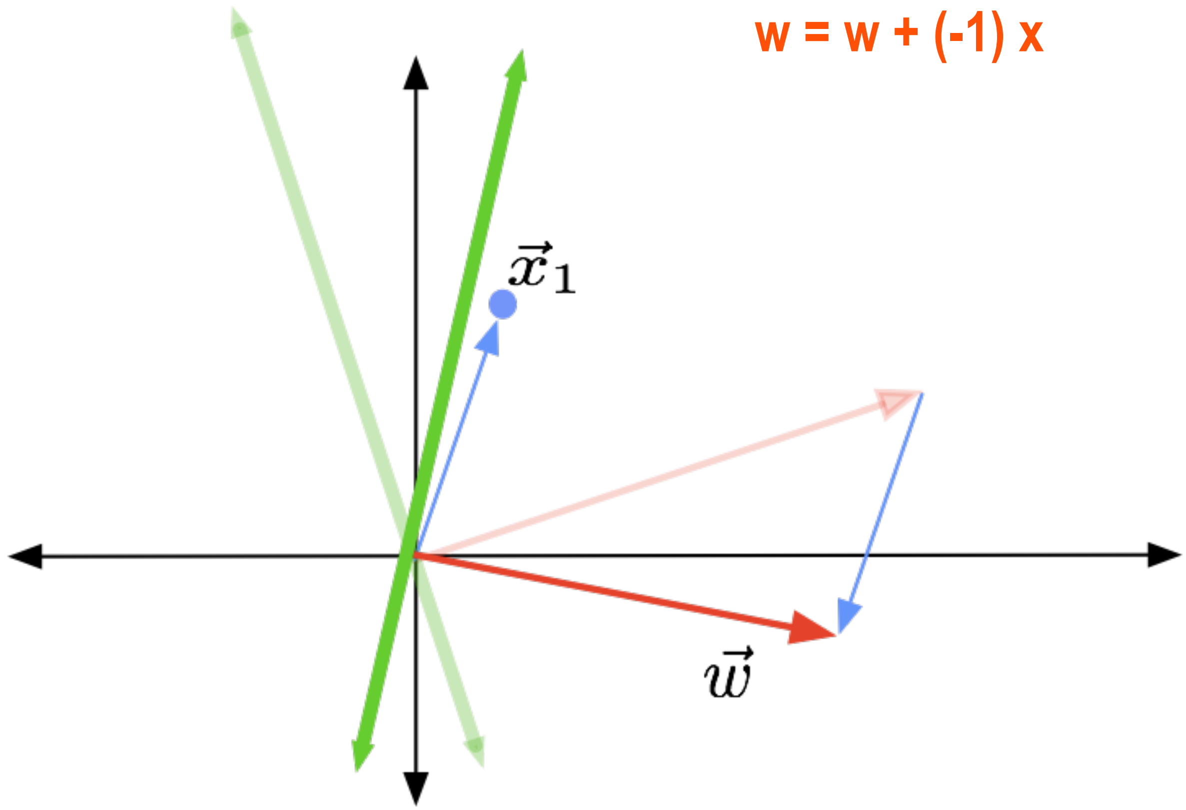


If the prediction at $\mathbf{x}_1$ is wrong, what is the true label y_1?

How do you update $\mathbf{w}$?

Perceptron update example

w = w + (-1) x



Properties of the simple perceptron

- **Provably:**
  - If it's possible to separate the data with a hyperplane (i.e. if it's linearly separable), then the algorithm will converge to that hyperplane.
  - And it will converge such that the number of mistakes M it makes is bounded by

    $$M < R^2/\gamma^2$$

    where

    $R = \max_i ||\mathbf{x}_i||_2$ size of biggest **x**

    $\gamma > y_i \; \mathbf{w}^{*T}\mathbf{x}_i$ > 0 if separable

Properties of the Simple Perceptron

But what if it isn't separable?

- Then perceptron is unstable and bounces around

Voted Perceptron

- Works just like a regular perceptron, except you keep track of all the intermediate models you created
- When you want to classify something, you let each of the many models *vote* on the answer and take the *majority*

Often implemented after a “burn-in” period

Properties of Voted Perceptron

- **Simple!**
- **Much better generalization performance than regular perceptron**
  - Almost as good as SVMs
  - Can use the 'kernel trick' – replace dot product with another kernel
- **Training is as fast as a regular perceptron**
- **But run-time is slower**
  - Since we need **n** models

Averaged Perceptron

- **The final model is the *average* of all the intermediate models**
- **Approximation to voted perceptron**
- **Again extremely simple!**
  - and can use kernels
- **Nearly as fast to train and exactly as fast to run as regular perceptron**

Many possible perceptrons

- **If point x_i is misclassified**
  - $\mathbf{w}_{i+1} = \mathbf{w}_i + \eta\, y_i\, \mathbf{x}_i$
- **Different ways of picking *learning rate* η**
- **Standard perceptron: $\eta = 1$**
  - Guaranteed to converge to the correct answer in a finite time if the points are separable (but oscillates otherwise)
  - Can get bounds on error even for non-separable case
- **Alternate: pick η to maximize the margin ($w_i^T x_i$) in some fashion**

Can we do a better job of picking η?

- **Perceptron:**

For each observation (y_i , $\mathbf{x}_i$)

$$\mathbf{w}_{i+1} = \mathbf{w}_i + \eta\, r_i\, \mathbf{x}_i$$

where $r_i = y_i - \text{sign}(\mathbf{w}_i^T\mathbf{x}_i)$

and $\eta = ½$

Let's use the fact that we are actually trying to minimize a loss function

Passive Aggressive Perceptron

- **Minimize the *hinge loss* at each observation**
  - $L(\mathbf{w}_i; \mathbf{x}_i, y_i) = 0$ if $y_i\, \mathbf{w}_i^T \mathbf{x}_i >= 1$ (loss 0 if correct with margin > 1)

    $1 - y_i\, \mathbf{w}_i^T \mathbf{x}_i$ else
- **Pick $\mathbf{w}_{i+1}$ to be as close as possible to $\mathbf{w}_i$ while still setting the hinge loss to zero**
  - If point x_i is correctly classified with a margin of at least 1
    - no change
  - Otherwise
    - $\mathbf{w}_{i+1} = \mathbf{w}_i + \eta\, y_i\, \mathbf{x}_i$
    - where $\eta = L(\mathbf{w}_i; \mathbf{x}_i, y_i)/||\mathbf{x}_i||^2$
- Can prove bounds on the total hinge loss

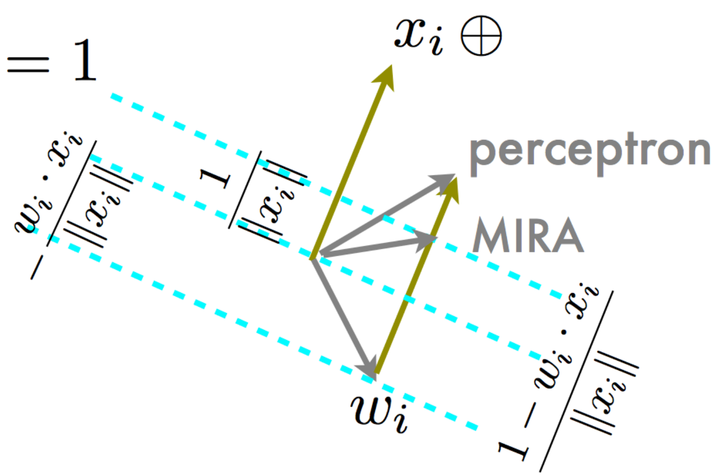
Passive-Aggressive = MIRA

$$w_{i+1} = w_i + \frac{y_i - w_i \cdot x_i}{\|x_i\|^2} x_i$$

easy to show:

$$y_i(w_{i+1} \cdot x_i) = y_i \; (w_i + \frac{y_i - w_i \cdot x_i}{\|x_i\|^2} x_i) \cdot x_i = 1$$

new score $\quad y_i \, (w_i * x_i \; + y_i - w_i * x_i) = y_i \, y_i$



Moves hyperplane so that new point is on the margin

Margin-Infused Relaxed Algorithm (MIRA)

- ***Multiclass*; each class has a prototype vector**
  - Note that the prototype *w* is like a feature vector *x*
- **Classify an instance by choosing the class whose prototype vector is *most similar* to the instance**
  - *Has the greatest dot product with the instance*
- **During training, make the 'smallest' change to the prototype vectors which guarantees correct classification by a specified margin**
  - "passive aggressive"

Can we parallelize SGD?

- If I give you 1,000 machines, how do you speed SDG up?

What we didn't cover

- Feature selection

What you should know

- **LMS**
  - Online regression
- **Perceptrons**
  - Online SVM
    - Large margin / hinge loss
  - Has nice mistake bounds (for separable case): see wiki
  - In practice, use averaged perceptrons
  - Passive Aggressive perceptrons and MIRA
    - Change *w* just enough to set its hinge loss to zero.