Generalized Linear Models (GLM) and Radial Basis Functions (RBF)

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Learning Objectives

Extend linear regression with link functions,
basis functions

Know RBF algorithm and its uses

Generalized linear models (GLM)

- **Linear Model:** $\hat{y}(\boldsymbol{x}) = \sum_{j=1}^{p} w_j x_j$
- **GLM with link fn $f()$:** $\hat{y}(\boldsymbol{x}) = f(\sum_{j=1}^{p} w_j x_j)$
- **Basis transformation:** $\hat{y}(\boldsymbol{x}) = \sum_{j=1}^{d} w_j \phi_j(\boldsymbol{x})$

Based on slide by Geoff Hinton

Link functions

- **Link function** $\hat{y} = f(\boldsymbol{w}^T\boldsymbol{x})$
  - $f(x) = e^x$
  - $f(x) = log(x)$
- **Equivalent to** $f^{-1}(\hat{y}(\boldsymbol{x})) = \boldsymbol{w}^T\boldsymbol{x}$

Linear Basis Function Models

- Generally,

$$\hat{y}(\boldsymbol{x}) = \sum_{j=1}^{d} w_j \underbrace{\phi_j(\boldsymbol{x})}_{\text{basis function}}$$

- Typically, $\phi_0(\boldsymbol{x}) = 1$ so that θ_0 w_0 acts as a bias
- In the simplest case, we use linear basis functions

$$\phi_j(\boldsymbol{x}) = x_j$$

- Could use polynomials or Gaussians

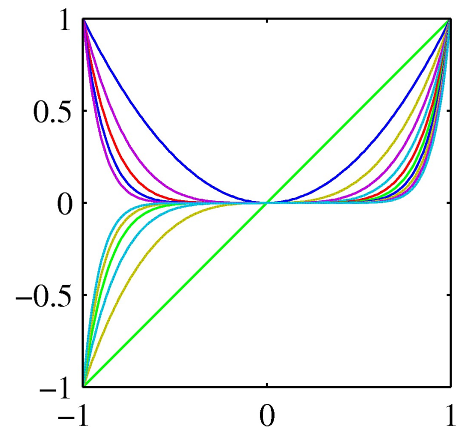
Based on slide by Christopher Bishop (PRML)

Linear Basis Function Models

- **Polynomial basis functions**

$$\phi_j(x) = x^j$$

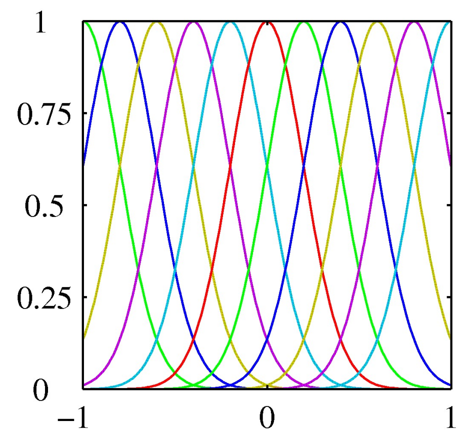
  - **Global – mostly crappy**



- **Gaussian basis functions:**

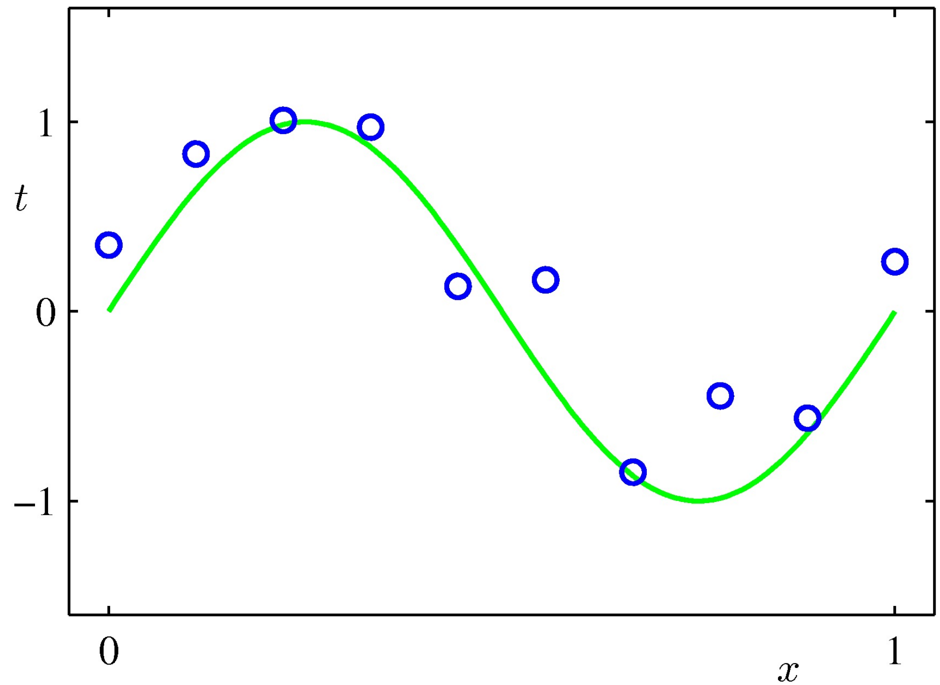
$$\phi_j(x) = \exp\left\{-\frac{(x-\mu_j)^2}{2s^2}\right\}$$

  - **Local – good!**



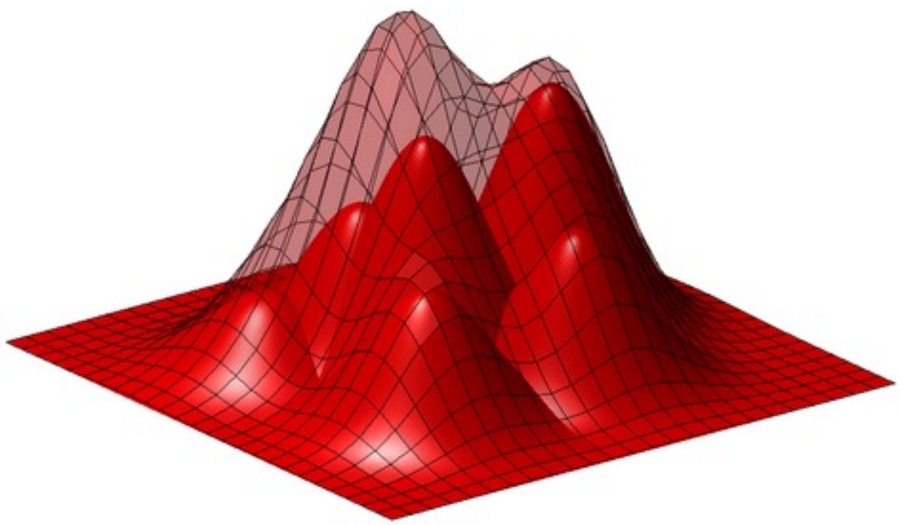
Based on slide by Christopher Bishop (PRML)

Fitting a Polynomial Curve with a Linear Model



$$y = \theta_0 + \theta_1 x + \theta_2 x^2 + \ldots + \theta_p x^p = \sum_{j=0}^{p} \theta_j x^j$$

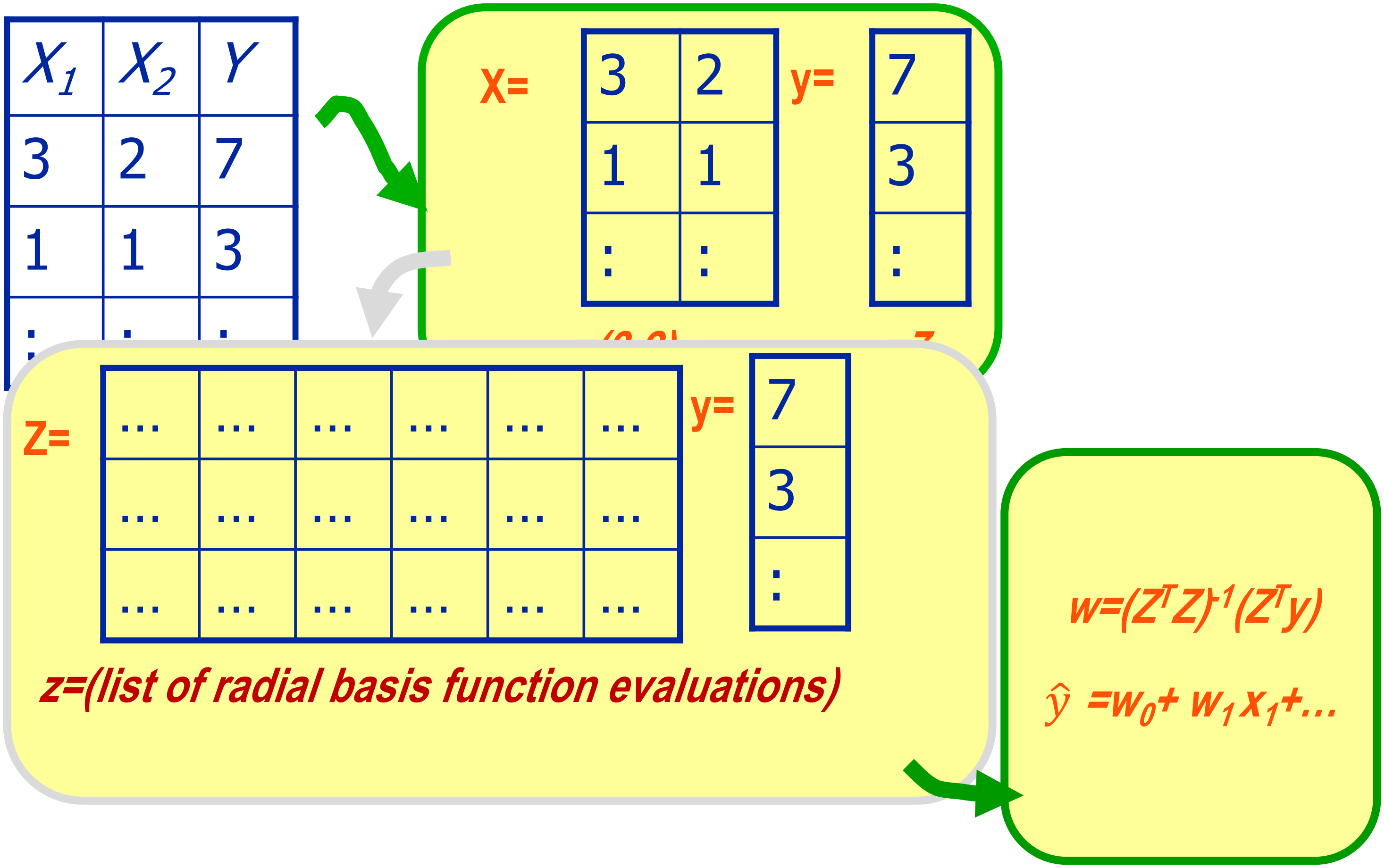
Radial Basis Functions



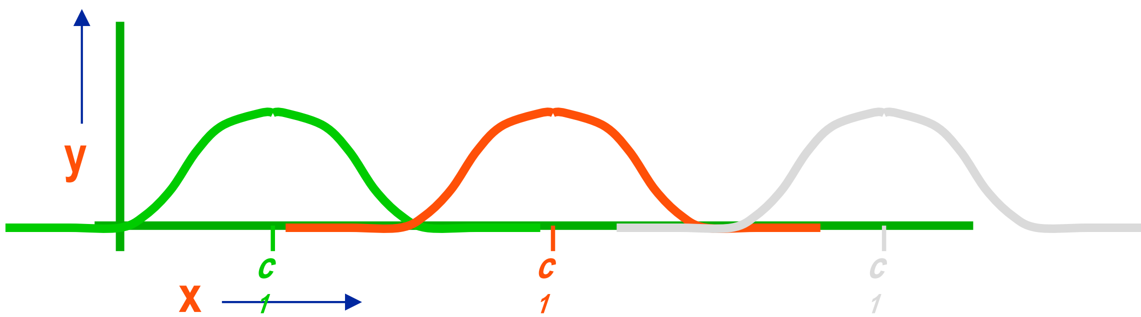
http://www.it.uu.se/research/project/rbf/rbf.png

Originally by Andrew Moore;
now heavily edited by Lyle Ungar

Radial Basis Functions (RBFs)



1-D RBFs



$\hat{y} = w_1 \phi_1(x) + w_2 \phi_2(x) + w_3 \phi_3(x)$

where

$\phi_i(x) = k(\|x - \mu_j\| / C)$

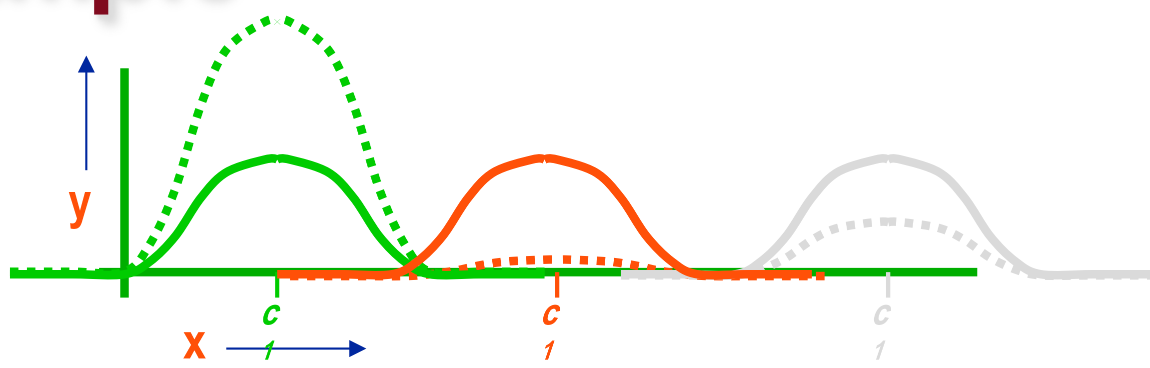
For RBF:

$k(\| x - \mu_j\| / C) = \exp\{-\|x - \mu_j\|_2^2 / C\}$

C = "Kernel Width"

k = kernel function

Example



$\hat{y} = 2\phi_1(x) + 0.05\phi_2(x) + 0.5\phi_3(x)$

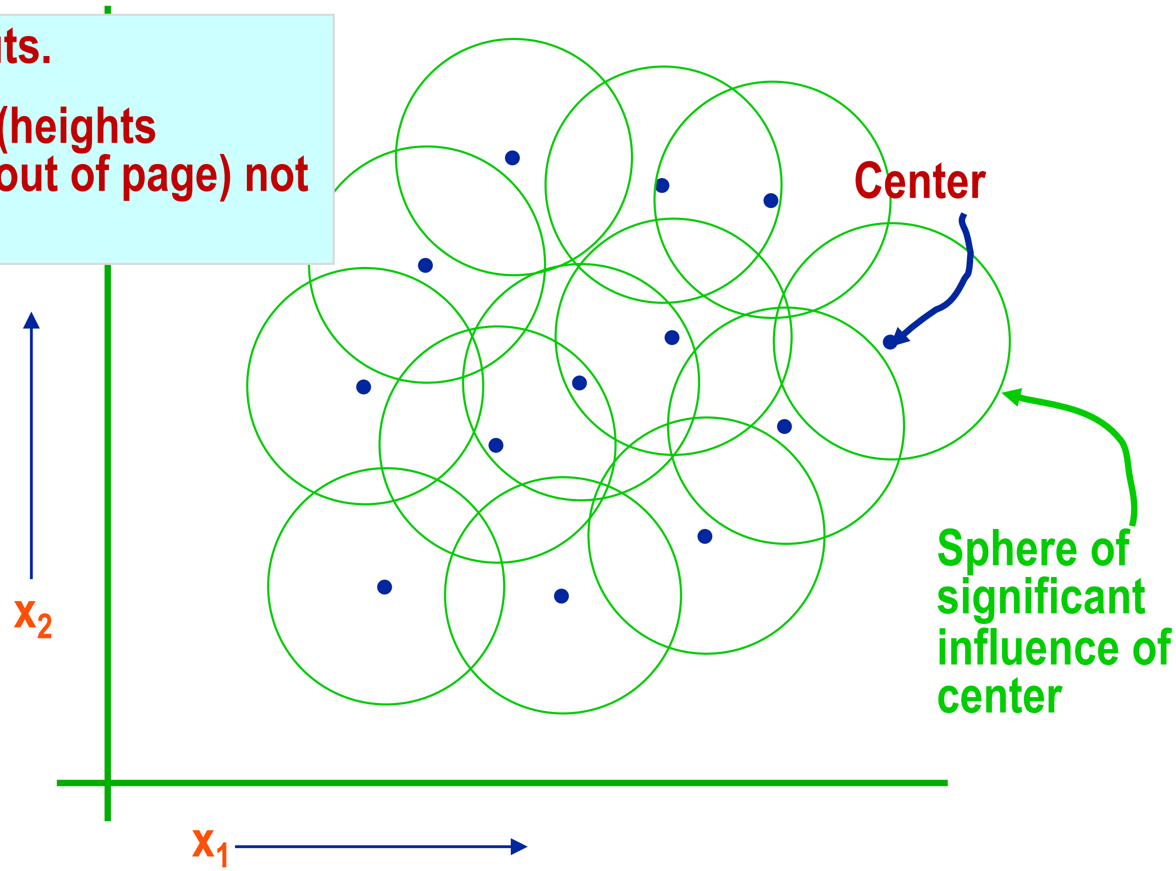
where

$\phi_j(x) = k(|| x - \mu_j|| / C)$

Radial Basis Functions in 2-d

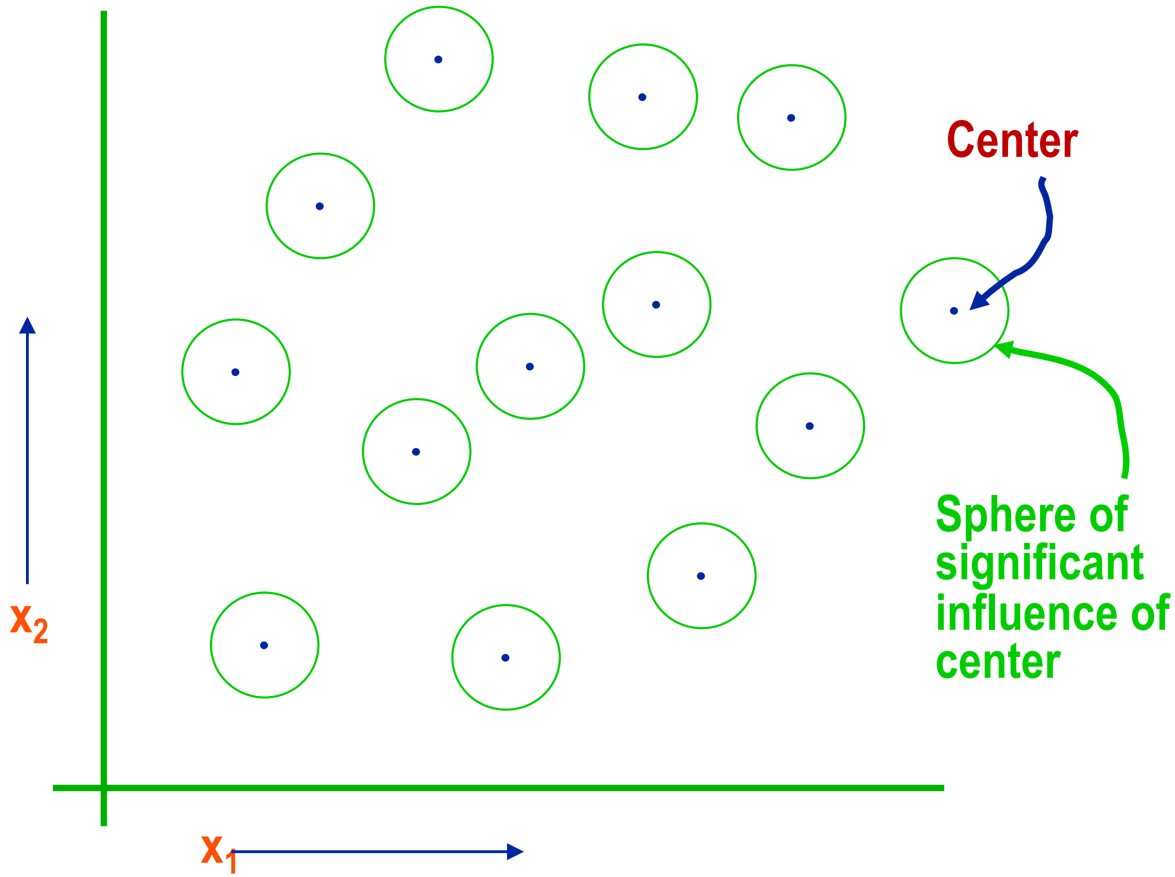
Two inputs.

Outputs (heights sticking out of page) not shown.

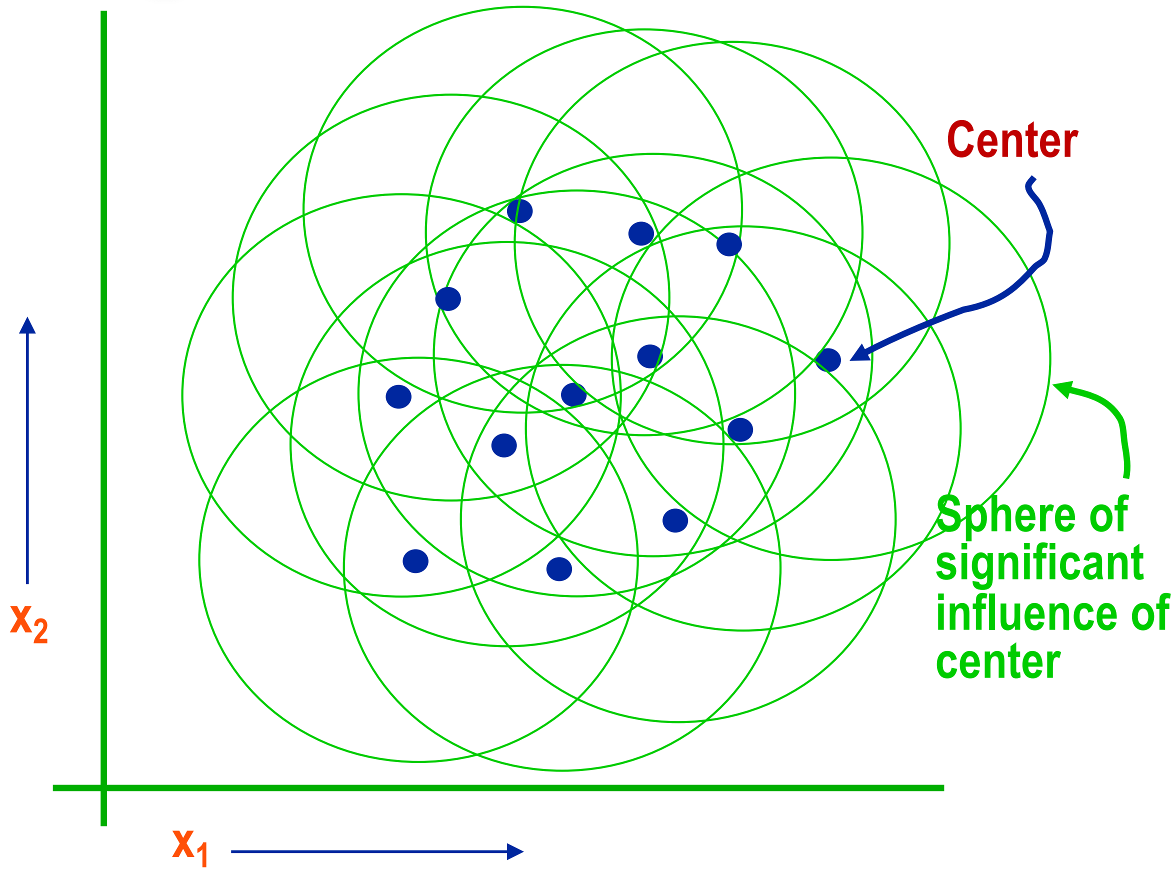


Too small!

Even before seeing the data, you should understand that this is a disaster!



Too big!!!



So what do we do?

Search to find the optimal size "width" for the Gaussians (on a test set, of course!)

RBFs can do …

- **Use $d < p$ basis vectors**
  - Dimensionality reduction
  - Good for high dimensional feature spaces
- **Use $d > p$ basis vectors**
  - Increases the dimensionality
  - Can make a formerly nonlinear problem linear
- **Use $d = n$ basis vectors**
  - We can use this to switch to a *dual* representation

How to find the kernel centers?

- **Pick random points**
  - Generally a bad idea
- **Standard RBF: do k-means clustering and use the centers of the clusters**
  - Works great!
- **Use all n of the training data points as kernel centers**
  - Requires regularization
- **Estimate them: nonlinear regression**
  - A good initialization helps

What you should know

- **Link functions give a nonlinear regression**
- **Basis functions allow one to fit a nonlinear function using linear regression**
- **RBF**
  - Cluster points
  - Put a Gaussian basic function at each cluster center
    - Pick the Gaussian width
  - Fit a linear regression

How is my speed?