Probabilistic and Bayesian Analytics

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Significantly modified by Lyle Ungar

What we're going to do

- **We will review the fundamentals of probability.**
- **It's really going to be worth it**
  - Much of this course builds on probabilities
  - E.g. Naïve Bayes, Logistic regression, Bayes nets, LDA …

Key concepts

- **Sample spaces, Events and Random Variables**
- **Expectation**
- **Probability distributions**
  - Discrete, continuous and joint distributions
  - Marginalization
  - PDFs and CDFs
- **Rules of probability**
  - Conditional probability, Bayes rule, Chain rule
  - Independence, Conditional independence

Discrete Random Variables

- **A is a Boolean-valued *random variable* if A denotes an event, and there is some degree of uncertainty as to whether A occurs.**
- **Examples**
- **A = The US president in 2023 will be male**
- **A = You wake up tomorrow with a headache**
- **A = You have Ebola**

How is this slide wrong?

A Random variable maps from an element of the sample space to a real number

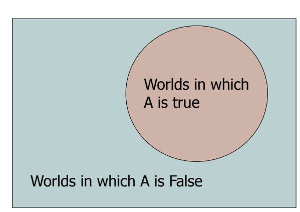
Probabilities

- **We write P(A) as "the fraction of possible worlds in which A is true"**
- **We could at this point spend 2 hours on the philosophy of this.**
- **But we won't.**

Visualizing A

Event space of all possible worlds

Its area is 1



P(A) = Area of reddish oval

The Axioms of Probability

- 0 <= P(A) <= 1
- P(True) = 1
- P(False) = 0
- P(A or B) = P(A) + P(B) - P(A and B)

Where do these axioms come from? Were they "discovered"? Answers coming up later.

Interpreting the axioms

- 0 <= P(A) <= 1
- P(True) = 1
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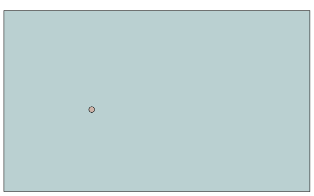
- P(A or B) = P(A) + P(B) - P(A and B)



The area of A can't get any smaller than 0

And a zero area would mean no world could ever have A true

Interpreting the axioms

- 0 <= P(A) <= 1
- P(True) = 1
- P(False) = 0
- P(A or B) = P(A) + P(B) - P(A and B)

The area of A can't get any bigger than 1

And an area of 1 would mean all worlds will have A true

Interpreting the axioms

- 0 <= P(A) <= 1
- P(True) = 1
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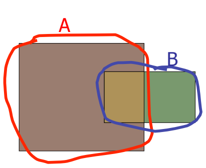

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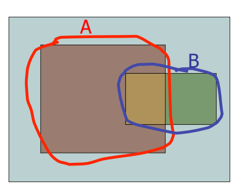

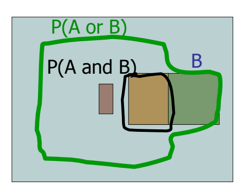

Simple addition and subtraction

These Axioms are Not to be Trifled With

- **There have been attempts to do different methodologies for uncertainty**
  - Fuzzy Logic
  - Three-valued logic
  - Dempster-Shafer
  - Non-monotonic reasoning
- **But the axioms of probability are the only system with this property:**

If you gamble using them you can't be unfairly exploited by an opponent using some other system [di Finetti 1931]

Theorems from the Axioms

- **0 <= P(A) <= 1,**
- **P(True) = 1,**
- **P(False) = 0**
- **P(A or B) = P(A) + P(B) - P(A and B)**

From these we can prove:

P(not A) = P(~A) = 1-P(A)

Side Note

- **I am inflicting these proofs on you for two reasons:**
  1. These kind of manipulations will need to be second nature to you if you use probabilistic analytics in depth
  2. Suffering is good for you

Another important theorem

- **0 <= P(A) <= 1,**
- **P(True) = 1,**
- **P(False) = 0**
- **P(A or B) = P(A) + P(B) - P(A and B)**

From these we can prove:

P(A) = P(A ^ B) + P(A ^ ~B)

Random Variables

- **A *random variable* maps from an element of a sample space to a discrete or real property of that element**
- **Examples**
  - S(x): person x is male → 1
    x is female → 0
  - A(x): = person x → x's age

Technically: random variable maps from an element of the sample space to a real number

We assign a probability to each outcome

- P(S(x) = 1)
- P(A(x) = 25)

Multivalued Random Variables

- **Suppose A can take on more than 2 values**
- **A is a *random variable with arity k* if it can take on exactly one value out of $\{v_1, v_2, .. v_k\}$**
- **Thus…**

$$P(A = v_i \wedge A = v_j) = 0 \text{ if } i \neq j$$
$$P(A = v_1 \vee A = v_2 \vee A = v_k) = 1$$

An easy fact about Multivalued Random Variables:

- **Using the axioms of probability…**
  0 <= P(A) <= 1, P(True) = 1, P(False) = 0
  P(A or B) = P(A) + P(B) - P(A and B)
- **And assuming that A obeys…**

$$P(A = v_i \wedge A = v_j) = 0 \text{ if } i \neq j$$

$$P(A = v_1 \vee A = v_2 \vee A = v_k) = 1$$

- **It's easy to prove that**

$$P(A = v_1 \vee A = v_2 \vee A = v_i) = \sum_{j=1}^{i} P(A = v_j)$$

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Another fact about Multivalued Random Variables:

- **Using the axioms of probability…**
  0 <= P(A) <= 1, P(True) = 1, P(False) = 0
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- It's easy to prove that

$$P(B \wedge [A = v_1 \vee A = v_2 \vee A = v_i]) = \sum_{j=1}^{i} P(B \wedge A = v_j)$$

Another fact about Multivalued Random Variables:

- **Using the axioms of probability…**
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- It' s easy to prove that

$$P(B \wedge [A = v_1 \vee A = v_2 \vee A = v_i]) = \sum_{j=1}^{i} P(B \wedge A = v_j)$$

- And thus we can prove

$$P(B) = \sum_{j=1}^{k} P(B \wedge A = v_j)$$

Elementary Probability in Pictures

- **P(~A) + P(A) = 1**

Elementary Probability in Pictures

- **P(B) = P(B ^ A) + P(B ^ ~A)**

Elementary Probability in Pictures

$$\sum_{j=1}^{k} P(A = v_j) = 1$$

Elementary Probability in Pictures

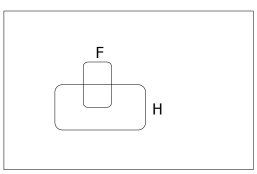
$$P(B) = \sum_{j=1}^{k} P(B \wedge A = v_j)$$

Conditional Probability

- **P(A|B) = Fraction of worlds in which B is true that also have A true**

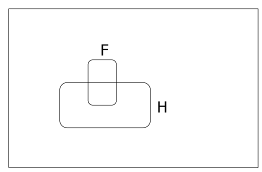
H = "Have a headache"
F = "Coming down with Flu"

P(H) = 1/10
P(F) = 1/40
P(H|F) = 1/2



"Headaches are rare and flu is rarer, but if you' re coming down with 'flu there' s a 50-50 chance you'll have a headache."

Conditional Probability



H = "Have a headache"
F = "Coming down with Flu"

P(H) = 1/10
P(F) = 1/40
P(H|F) = 1/2

P(H|F) = Fraction of flu-inflicted worlds in which you have a headache

= #worlds with flu and headache

#worlds with flu

= Area of "H and F" region

Area of "F" region

= P(H ^ F)

P(F)

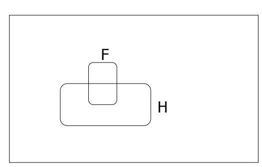
Definition of Conditional Probability

$$P(A|B) = \frac{P(A \wedge B)}{P(B)}$$

Corollary: The Chain Rule

$$P(A \wedge B) = P(A|B)\, P(B)$$

Probabilistic Inference



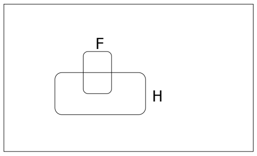
H = "Have a headache"
F = "Coming down with Flu"

P(H) = 1/10
P(F) = 1/40
P(H|F) = 1/2

One day you wake up with a headache. You think: "Drat! 50% of flus are associated with headaches so I must have a 50-50 chance of coming down with flu"

Is this reasoning good?

Probabilistic Inference



H = "Have a headache"
F = "Coming down with Flu"

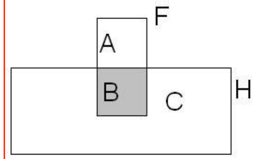
P(H) = 1/10
P(F) = 1/40
P(H|F) = 1/2

P(F ^ H) = ...

P(F|H) = ...

Another way to understand the intuition

Thanks to Jahanzeb Sherwani for contributing this explanation:



Let's say we have P(F), P(H), and P(H|F), like in the example in class.

Areawise, P(F) = A + B, P(H) = B + C,

Also, P(H|F) = $\frac{B}{A + B}$

Thus, to get the opposite conditional probability, ie, P(F|H), we need to figure out $\frac{B}{B + C}$

Since we know B / (A+B), we can get B / (B+C) by multiplying by (A+B) and dividing by (B+C). But since we already calculated, A+B = P(F), and B+C = P(H), so we are actually multiplying by P(F) and dividing by P(H). Which is Bayes Rule:

P(F|H) = P(H|F) * $\frac{P(F)}{P(H)}$

What we just did…

$$P(B|A) = \frac{P(A \wedge B)}{P(A)} = \frac{P(A|B)\,P(B)}{P(A)}$$

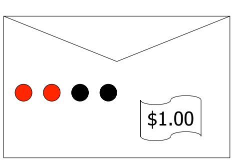
This is Bayes Rule

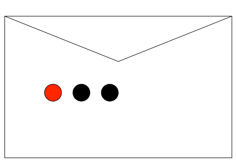
Bayes, Thomas (1763) An essay towards solving a problem in the doctrine of chances. *Philosophical Transactions of the Royal Society of London,* **53:370-418**

Bayes rule made easy

- **B** **~ B**
- **A** 0.01 0.1
- **~A** 0.09 0.8
- **Counts instead of probabilities**
- **B** **~ B**
- **A** 1 10
- **~A** 9 80

Using Bayes Rule to Gamble



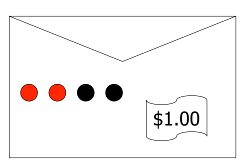


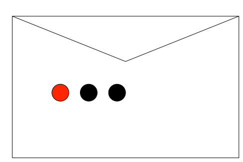
The "Win" envelope has a dollar and four beads in it

The "Lose" envelope has three beads and no money

Trivial question: someone draws an envelope at random and offers to sell it to you. How much should you pay?

Using Bayes Rule to Gamble





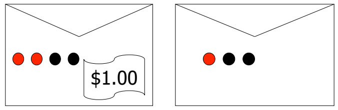
The "Win" envelope has a dollar and four beads in it

The "Lose" envelope has three beads and no money

Interesting question: before deciding, you are allowed to see one bead drawn from the envelope.

Suppose it's black: How much should you pay?
Suppose it's red: How much should you pay?

Calculation…



More General Forms of Bayes Rule

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|\sim A)P(\sim A)}$$

$$P(A|B \wedge X) = \frac{P(B|A \wedge X)P(A \wedge X)}{P(B \wedge X)}$$

More General Forms of Bayes Rule

$$P(A = v_i|B) = \frac{P(B|A = v_i)P(A = v_i)}{\sum_{k=1}^{n_A} P(B|A = v_k)P(A = v_k)}$$

Useful Easy-to-prove facts

$$P(A|B) + P(\neg A|B) = 1$$

$$\sum_{k=1}^{n_A} P(A = v_k|B) = 1$$

The Joint Distribution

Example: Boolean variables A, B, C

Recipe for making a joint distribution of M variables:

The Joint Distribution

Example: Boolean variables A, B, C

Recipe for making a joint distribution of M variables:

1. Make a truth table listing all combinations of values of your variables (if there are M Boolean variables then the table will have 2^M rows).

A	B	C
0	0	0
0	0	1
0	1	0
0	1	1
1	0	0
1	0	1
1	1	0
1	1	1

The Joint Distribution

Example: Boolean variables A, B, C

A	B	C	Prob
0	0	0	0.30
0	0	1	0.05
0	1	0	0.10
0	1	1	0.05
1	0	0	0.05
1	0	1	0.10
1	1	0	0.25
1	1	1	0.10

Recipe for making a joint distribution of M variables:

1. Make a truth table listing all combinations of values of your variables (if there are M Boolean variables then the table will have 2^M rows).
2. For each combination of values, say how probable it is.

The Joint Distribution

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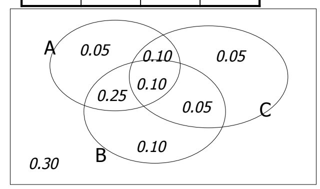
Recipe for making a joint distribution of M variables:

1. Make a truth table listing all combinations of values of your variables (if there are M Boolean variables then the table will have 2^M rows).
2. For each combination of values, say how probable it is.
3. If you subscribe to the axioms of probability, those numbers must sum to 1.



Using the Joint

gender	hours_worked	wealth	
Female	v0:40.5-	poor	0.253122
		rich	0.0245895
	v1:40.5+	poor	0.0421768
		rich	0.0116293
Male	v0:40.5-	poor	0.331313
		rich	0.0971295
	v1:40.5+	poor	0.134106
		rich	0.105933

One you have the JD you can ask for the probability of any logical expression involving your attribute

$$P(E) = \sum_{\text{rows matching } E} P(\text{row})$$

Using the Joint

gender	hours_worked	wealth	
Female	v0:40.5-	poor	0.253122
		rich	0.0245895
	v1:40.5+	poor	0.0421768
		rich	0.0116293
Male	v0:40.5-	poor	0.331313
		rich	0.0971295
	v1:40.5+	poor	0.134106
		rich	0.105933

P(Poor Male) = 0.4654

$$P(E) = \sum_{\text{rows matching } E} P(\text{row})$$

Using the Joint

gender	hours_worked	wealth	
Female	v0:40.5-	poor	0.253122
		rich	0.0245895
	v1:40.5+	poor	0.0421768
		rich	0.0116293
Male	v0:40.5-	poor	0.331313
		rich	0.0971295
	v1:40.5+	poor	0.134106
		rich	0.105933

P(Poor) = 0.7604

$$P(E) = \sum_{\text{rows matching } E} P(\text{row})$$

Inference with the Joint

gender	hours_worked	wealth	
Female	v0:40.5-	poor	0.253122
		rich	0.0245895
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$$P(E_1 \mid E_2) = \frac{P(E_1 \wedge E_2)}{P(E_2)} = \frac{\sum_{\text{rows matching } E_1 \text{ and } E_2} P(\text{row})}{\sum_{\text{rows matching } E_2} P(\text{row})}$$

Inference with the Joint

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$$P(E_1 | E_2) = \frac{P(E_1 \wedge E_2)}{P(E_2)} = \frac{\sum_{\text{rows matching } E_1 \text{ and } E_2} P(\text{row})}{\sum_{\text{rows matching } E_2} P(\text{row})}$$

P(Male | Poor) = 0.4654 / 0.7604 = 0.612

Inference is a big deal

- **I've got this evidence. What's the chance that this conclusion is true?**
  - I've got a sore neck: how likely am I to have meningitis?
  - I see my lights are out and it's 9pm. What's the chance my spouse is already asleep?
- **There's a thriving set of industries growing based around Bayesian Inference, including**
  - Medicine, Pharma, Help Desk Support, Engine Fault Diagnosis

Where do Joint Distributions come from?

- **Idea One: Expert Humans**
- **Idea Two: Simpler probabilistic facts and some algebra**

Example: Suppose you knew

P(A) = 0.7
P(B|A) = 0.2
P(B|~A) = 0.1

P(C|A^B) = 0.1
P(C|A^~B) = 0.8
P(C|~A^B) = 0.3
P(C|~A^~B) = 0.1

Then you can automatically compute the JD using the chain rule

P(A=x ^ B=y ^ C=z) =
P(C=z|A=x^ B=y) P(B=y|A=x) P(A=x)

In another lecture: Bayes Nets, a systematic way to do this.

Where do Joint Distributions come from?

- **Idea Three: Learn them from data!**

Prepare to see one of the most impressive learning algorithms you'll come across in the entire course….

Learning a joint distribution

Build a JD table for your attributes in which the probabilities are unspecified

A	B	C	Prob
0	0	0	?
0	0	1	?
0	1	0	?
0	1	1	?
1	0	0	?
1	0	1	?
1	1	0	?
1	1	1	?

The fill in each row with

$$\hat{P}(\text{row}) = \frac{\text{records matching row}}{\text{total number of records}}$$

A	B	C	Prob
0	0	0	0.30
0	0	1	0.05
0	1	0	0.10
0	1	1	0.05
1	0	0	0.05
1	0	1	0.10
1	1	0	0.25
1	1	1	0.10

Fraction of all records in which A and B are True but C is False

Example of Learning a Joint

- This Joint was obtained by learning from three attributes in the UCI "Adult" Census Database [Kohavi 1995]

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Female	v0:40.5-	poor	0.253122
		rich	0.0245895
	v1:40.5+	poor	0.0421768
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