Perceptrons, SVMs, and Friends:
Some *Discriminative* Models for Classification

Parallel to AIMA 18.1, 18.2, 18.6.3, 18.9
The Automatic Classification Problem

- Assign object/event or sequence of objects/events to one of a given finite set of categories.
  - *Fraud detection* for credit card transactions, telephone calls, etc.
  - *Worm detection* in network packets
  - *Spam filtering* in email
  - *Recommending articles*, books, movies, music
  - *Medical diagnosis*
  - *Speech recognition*
  - *OCR* of handwritten letters
  - Recognition of specific astronomical images
  - Recognition of specific DNA sequences
  - Financial investment

- Machine Learning methods provide one set of approaches to this problem
Universal Machine Learning Diagram

Things to be classified

Feature Vector Representation

Magic Classifier Box

Classification Decision
Example: handwritten digit recognition

Machine learning algorithms that

- Automatically cluster these images
- Use a training set of labeled images to learn to classify new images
- Discover how to account for variability in writing style
A machine learning algorithm development pipeline: minimization

Problem statement

Mathematical description of a cost function

Mathematical description of how to minimize the cost function

Implementation

Given training vectors $x_1, \ldots, x_N$ and targets $t_1, \ldots, t_N$, find...

$$E(w) = \mathcal{L}(\theta)$$

$$p(x|w)$$

$$\frac{\partial E}{\partial w_i}$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = 0$$

$$r(i,k) = s(i,k) - \max_j \{s(i,j) + a(i,j)\}$$

...
Today: Perceptron, SVM and Friends

- Naïve Bayes Classifiers are one example
Generative vs. Discriminative Models

- **Generative question:**
  - “How can we model the joint distribution of the classes and the features?”

\[
c_{MAP} \equiv \arg \max_{c \in C} P(c \mid D)
\]

\[
c_{ML} \equiv \arg \max_{c \in C} P(D \mid c)
\]

- Naïve Bayes, Markov Models, HMMs all generative

- **Discriminative question:**
  - “What features distinguish the classes from one another?”
Example

Modeling what sort of bizarre distribution produced these training points is hard, but distinguishing the classes is a piece of cake!

*chart from MIT tech report #507, Tony Jebara*
Linear Classification: Informal…

Find a (line, plane, hyperplane) that divides the red points from the blue points….
Hyperplane

A hyperplane can be defined by

\[ c = \mathbf{w} \cdot \mathbf{x} \]

Or more simply (renormalizing) by

\[ 0 = \mathbf{w} \cdot \mathbf{x} \]

Consider a two-dimension example…

\[ 0 = [1, -1] \begin{bmatrix} x \\ y \end{bmatrix} \]

\[ 0 = x - y \]

\[ y = x \]
Linear Classification: Slightly more formal

Input encoded as feature vector $\vec{x}$

Model encoded as $\vec{w}$

Just return $y = \vec{w} \cdot \vec{x}$!

$\text{sign}(y)$ tell us the class:
- $+$ - blue
- $-$ - red

(All vectors normalized to length 1, for simplicity)
Computing the sign...

One definition of dot product: 

\[ W \cdot X = \| W \| \| X \| \cos \theta \]

So \( \text{sign}(W \cdot X) = \text{sign}(\cos \theta) \)

Let \( y = \text{sign}(\cos \theta) \)
Perceptron Update Example

If \( \vec{x}_1 \) is supposed to be on the other side….

\[
\vec{w} = \vec{w} + y_i \vec{x}_i
\]
Perceptron Learning Algorithm

**Input:** A list $T$ of training examples $\langle \vec{x}_0, y_0 \rangle \ldots \langle \vec{x}_n, y_n \rangle$ where
\[
\forall i : y_i \in \{ +1, -1 \}
\]

**Output:** A classifying hyperplane $\vec{w}$

Randomly initialize $\vec{w}$;

**while** model $\vec{w}$ makes errors on the training data **do**

**for** $\langle \vec{x}_i, y_i \rangle$ **in** $T$ **do**

\[
\vec{W} = \vec{W} + y_i \vec{x}_i
\]

Let $\hat{y} = \text{sign}(\vec{w} \cdot \vec{x}_i)$;

**if** $\hat{y} \neq y_i$ **then**

\[
\vec{w} = \vec{w} + y_i \vec{x}_i;
\]

**end**

**end**

**end**

Converges if the training set is *linearly separable*

May not converge if the training set is *not* linearly separable
Compared to the biological neuron

- **Input**
  - A neuron's dendritic tree is connected to a thousand neighboring neurons. When one of those neurons fire, a positive or negative charge is received.
  - The strengths of all the received charges are added together …

- **Output**
  - If the aggregate input is greater than the axon hillock's threshold value, then the neuron fires.
  - The physical and neurochemical characteristics of each synapse determines the strength and polarity of the new signal.

\[ f(x) = \begin{cases} 
1 & \text{if } w \cdot x + b > 0 \\
0 & \text{else}
\end{cases} \]
Voted & Averaged Perceptron

--Works just like a regular perceptron, except keeping track of all the intermediate models created

--Much better generalization performance than regular perceptron (almost as good as SVMs)

• Voted Perceptron (Freund & Schapire 1999)
  • let each of the (many, many) models vote on the answer and take the majority
  • As fast to train but slower in run-time

• Averaged Perceptron (Collins 2002)
  • Return as your final model the average of all intermediate models
  • Nearly as fast to train and exactly as fast to run as regular perceptron
Properties of the Simple Perceptron

- You can prove that
  - If it’s possible to separate the data with a hyperplane (i.e. if it’s linearly separable),
  - Then the algorithm will converge to that hyperplane.

- But what if it isn’t? Then perceptron is very unstable and oscillates back and forth.
Support vector machines
What’s wrong with these hyperplanes?
They’re unjustifiably biased!
A less biased choice
Margin

- the distance to closest point in the training data
- We tend to get better generalization to **unseen data** if we choose the separating hyperplane which *maximizes the margin*
Support Vector Machines

- A learning method which *explicitly calculates the maximum margin hyperplane* by solving a gigantic quadratic programming minimization problem.
- Among the very highest-performing current machine learning techniques.
- But it’s relatively slow and quite complicated.
Maximizing the Margin

Select the separating hyperplane that maximizes the margin.
Support Vectors

Margin Width

Support Vectors

\( x_1 \)

\( x_2 \)
Support Vector Machines

- A learning method which *explicitly calculates the maximum margin hyperplane.*
Setting Up the Optimization Problem

The maximum margin can be characterized as a solution to an optimization problem:

\[
\begin{align*}
\text{max.} & \quad \frac{2}{\|w\|} \\
\text{s.t.} & \quad (w \cdot x + b) \geq 1, \quad \forall x \text{ of class 1} \\
& \quad (w \cdot x + b) \leq -1, \quad \forall x \text{ of class 2}
\end{align*}
\]

Define the margin (what ever it turns out to be) to be one unit of width.
Setting Up the Optimization Problem

• If class 1 corresponds to 1 and class 2 corresponds to -1, we can rewrite
  \( (w \cdot x_i + b) \geq 1, \ \forall x_i \) with \( y_i = 1 \)
  \( (w \cdot x_i + b) \leq -1, \ \forall x_i \) with \( y_i = -1 \)

• as
  \[ y_i (w \cdot x_i + b) \geq 1, \ \forall x_i \]

• So the problem becomes:
  \[
  \text{max. } \frac{2}{\|w\|} \quad \text{or} \quad \text{min. } \frac{1}{2} \|w\|^2
  \]
  \[ s.t. \ y_i (w \cdot x_i + b) \geq 1, \ \forall x_i \quad s.t. \ y_i (w \cdot x_i + b) \geq 1, \ \forall x_i \]
Linear, (Hard-Margin) SVM Formulation

- Find \( w, b \) that solves

\[
\begin{align*}
\text{min.} & \quad \frac{1}{2} \| w \|^2 \\
\text{s.t.} & \quad y_i (w \cdot x_i + b) \geq 1, \quad \forall x_i
\end{align*}
\]

- Problem is \textit{convex}, so there is a unique global minimum value (when feasible)
- There is also a unique minimizer, i.e. weight and \( b \) value that provides the minimum
- Quadratic Programming
  - very efficient computationally with procedures that take advantage of the special structure
What if it isn’t separable?
Project it to someplace where it is!

\[ \phi(\langle x, y \rangle) = x^2 + y^2 \]
Non-linear SVMs: Feature spaces

- General idea: the original feature space can always be mapped to some higher-dimensional feature space where the training set is linearly separable:

\[ \Phi: x \rightarrow \varphi(x) \]
Kernel Trick

- If our data isn’t linearly separable, we can define a projection $\Phi(x_i)$ to map it into a much higher dimensional feature space where it is.

- For SVM where everything can be expressed as the dot products of instances this can be done efficiently using the `kernel trick’:
  - A kernel $K$ is a function such that: $K(x_i, x_j) = \Phi(x_i) \cdot \Phi(x_j)$
  - Then, we never need to explicitly map the data into the high-dimensional space to solve the optimization problem – magic!!
Gaussian Kernel: Example

The appropriate K maps this into a hyperplane in some space!!
SVMs vs. other ML methods

- Examples from the NIST database of handwritten digits
  - 60K labeled digits 20x20 pixels 8bit greyscale values
- Learning methods
  - 3-nearest neighbors
  - Hidden layer neural net
  - Specialized neural net (LeNet)
  - Boosted neural net
  - SVM
  - SVM with kernels on pairs of nearby pixels + specialized transforms
  - Shape matching (vision technique)
- Human error: on similar US Post Office database 2.5%.
## Performance on the NIST digit set (2003)

<table>
<thead>
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<th>3-NN</th>
<th>Hidden Layer NN</th>
<th>LeNet</th>
<th>Boosted LeNet</th>
<th>SVM</th>
<th>Kernel SVM</th>
<th>Shape Match</th>
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<tr>
<td>Error %</td>
<td>2.4</td>
<td>1.6</td>
<td>0.9</td>
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<td>Run time (millisec/digit)</td>
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<td>50</td>
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<td>.21</td>
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<td>Training time (days)</td>
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<td>7</td>
<td>14</td>
<td>30</td>
<td>10</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Recently beaten (2010) (.35% error) by a very complex neural network (if you want details: a 6 layer NN with 784-2500-2000-1500-1000-500-10 topology with elastic distortions running on modern GPU)