1 Short questions  [25 points]

1. [5 points] Derivative filters When we need to take a first derivative, we generally use the discrete filter \([1/2 \ 0 -1/2]\), and we use \([1 -2 1]\) for the second derivative. Why don’t we use \([1/2\ 0 -1/2]\) convolved with itself for the second derivative? Analytically compute and plot the Fourier Transform of \([1 -2 1]\), \([1/2\ 0 -1/2]\) convolved with itself, and the Fourier Transform of the theoretical 2nd-order derivative, \(\frac{d^2}{dt^2}\). Conclude.

2. The step function is defined as
   \[ h(t) = \begin{cases} 1, & \text{if } t \geq 0 \\ 0, & \text{otherwise.} \end{cases} \]

   (a) [1 point] Compute the convolution of \(h(t)\) with \(g'(t)\).

   (b) [2 points] A double-step of width \(a\) is defined as \(d(t) = h(t) + h(t-a)\). Compute the convolution \((d \star g')(t)\).

   (c) [3 points] An edge can be defined as the local maximum of \((d \star g')(t)\). Compute the local maxima of \((d \star g')(t)\). Heuristics: You might use manual heuristics as well as some plot based arguments to solve the underlying transcendental equation.

   (d) [4 points] Establish the relation between \(\sigma\) and step-width \(a\) which will change the number of local maxima (and thus detected edges) of \((d \star g')(t)\).

3. Binomial filters The binomial filters are defined as follows:
   \[ B_N = (([11] \star \ldots \star [11]))_{N \text{ times}} \]

   (a) [4 points] Compute the Fourier transform of \(B_N\).

   (b) [4 points] Show that as \(N\) increases, \(B_N\) approximates a Gaussian of \(\sigma = \frac{\sqrt{N}}{2}\) better and better.

   (c) [2 points] What is the interest of having such an approximation?
2 Image sampling and interpolation  [28 points]

In this section we will subsample an image, and we will then try to reconstruct the original image by performing 2D interpolation. Figure 1 shows the subsampling every $T_s$ of $I$ to obtain $I_{\text{down}} (T_s = 4$ here), and the reconstruction process for an image that was subsampled at $T_s$: sampled values are placed on a grid of step $T_s$ and an interpolation function (typically sinc or $\text{sinc}^2$) is convolved with the grid image to obtain the interpolated image.

![Images showing subsampling and reconstruction](image.png)

Figure 1: Subsampling (1b) and reconstruction by interpolation (1c and 1d).

1. Sinc interpolation

(a) [4 points] Draw a diagram to explain (1) what happens in the frequency domain when we sample a signal every $T_s$, and (2) what happens in the time domain when we box the spectrum (multiply by a rectangle $H(f) = 1(|f| < f_s/2)$ where $f_s = 1/T_s$) to reconstruct the signal.

(b) [3 points] Prove that if we subsample a 1D signal at $T_s$ and reconstruct it as in question 1a, it corresponds to convolving the the sampled grid (third image in figure 1) with the interpolation function $\text{sinc} \left( \frac{\pi}{T_s} \right)$ where sinc is the unnormalized sinc function. (Note that in MATLAB, $\text{sinc}(x)$ is the normalized sinc function, sometimes denoted as $\text{sinc}_\pi(x) \equiv \text{sinc}(\pi x)$.)

(c) [2 points] For a 2D image, we box the 2D spectrum with a 2D rectangle function (still between $-f_s/2$ and $f_s/2$ along $f_x$ and $f_y$). What is the corresponding 2D interpolation function? You can write your answer as the convolution of two 1D functions, one along $x$ and the other along $y$.

(d) [10 points] Write a function that:

- takes a grayscale image as input,
- subsamples it at $T_s = 4$,
- places the subsampled values every $T_s$ (along $x$ and $y$) in a image that is originally 0 everywhere,
- convolves the above “grid image” with the 2D interpolation function computed in question 1c.

Explain the aliasing you observe, and compute the SNR of the reconstruction with respect to the original image:

$$\text{SNR} = -20 \log \left( \frac{||I - I_{\text{reconstr}}||_2}{||I||_2} \right),$$

where $||I||_2$ is the square root of the sum of all squared intensities in image $I$.

**Your image approximation should always look like the original image up to a few artifacts, otherwise it means you are doing something wrong.**

2. Anti-aliasing

You probably noticed some aliasing or ringing artifacts in the reconstructed image. To avoid them, one can filter the original image before subsampling it.

(a) [4 points] With a figure in the Fourier domain, explain why smoothing the signal before subsampling reduces the risk of aliasing.

(b) [3 points] Add a Gaussian smoothing step before the subsampling in your code and check that it helps with aliasing. Is the SNR with respect to the original (non-smoothed) image better? Why?

(c) [2 points] Cite another simple way of avoiding aliasing.