# *CIS 580 Spring 2012 - Lecture 1*

*January 11, 2012*

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## *Linear Shift-Invariant Systems*

Consider a continuous signal $f : \mathbb{R} \to \mathbb{R},\ t \to f(t)$ and a filter $f(t) \to g(t) = T\{f(t)\}$:

Figure 1: A filter represented as a block

$$f(t) \longrightarrow \boxed{\text{System } T} \longrightarrow g(t)$$

### *Linear system*

A system $T$ is **linear** when $T\{af_1(t) + bf_2(t)\} = aT\{f_1(t)\} + bT\{f_2(t)\}$:

Figure 2: Linear System

$$\begin{array}{l} f_1(t) \longrightarrow \boxed{T} \longrightarrow g_1(t) \\ f_2(t) \longrightarrow \boxed{T} \longrightarrow g_2(t) \end{array} \quad \text{then} \quad af_1(t) + bf_2(t) \longrightarrow \boxed{T} \longrightarrow ag_1(t) + bg_2(t)$$

Examples:

- $T\{f\}(t) = f(t) - f(t-1)$ is linear:

$$\begin{aligned} g_1(t) &= f_1(t) - f_1(t-1) \\ g_2(t) &= f_2(t) - f_2(t-1) \\ T\{af_1 + bf_2\}(t) &= [af_1(t) + bf_2(t)] - [af_1(t-1) - bf_2(t-1)] \\ &= a(f_1(t) - f_1(t-1)) + b(f_2(t) - f_2(t-1)) \\ &= aT\{f_1\}(t) + bT\{f_2\}(t) \end{aligned}$$

- $g(t) = f(t)^2$, $g(t) = \max(f(t), f(t-1))$ are not linear.

### *Shift invariant system*

A system is **shift-invariant** when $T\{f(t-t_0)\}(t) = T\{f(t)\}(t-t_0)$:

Figure 3: Shift-Invariant System

$$f(t) \longrightarrow \boxed{T} \longrightarrow g(t) \quad \text{then} \quad f(t-t_0) \longrightarrow \boxed{T} \longrightarrow g(t-t_0)$$

Examples:

- $T\{f\}(t) = f(t) - f(t-1)$ is shift-invariant.

$$\begin{aligned} T\{f(t-t_0)\} =& f(t-t_0) - f(t-t_0-1) \\ =& g(t-t_0) \end{aligned}$$

- $g(t) = tf(t)$ is not shift-invariant.

*Dirac function and impulse response*

Definitions of the Dirac function:

1. $\delta(t) = \lim_{a \to 0} \frac{1}{a} \mathrm{rect}(\frac{t}{a})$,

$$\mathrm{rect}(t) = \begin{cases} 1 & |t| \leq 1/2 \\ 0 & \text{elsewhere} \end{cases}$$

2. Absorption property $\int_{-\infty}^{\infty} f(t)\delta(t)dt = f(0)$
3. $\delta(t) = 0$ for $t \neq 0$ and $\int_{-\infty}^{\infty} \delta(t)dt = 1$

The *impulse response* of a system $T$ is the output of the system when the input is a Dirac:

$$h(t) = \int_{-\infty}^{\infty} \delta(t')h(t-t')dt',$$

holds because of the absorption property $t \to t - t'$.

*LSI as a convolution*

Question: is there a formula describing the action of a general LSI system?

Answer: Yes, it is the convolution of the signal with the *impulse response* $h(t)$:

$$g(t) = \int_{-\infty}^{\infty} f(t')h(t-t')dt'$$

.

Notice that $h(t-t')$ is a reflection and shift of the impulse response (see figure 4). To understand why we take the reflection of the impulse response, we consider a special type of LSI's: **causal** systems, where $g(t)$ only depends on values of $f(t')$ for $t' \leq t$. This means the impulse response will be non-zero for $t \geq 0$.

For example let's consider $T : f(t) \to T\{f(t)\} = g(t) = f(t) + f(t-1) + f(t-2)$. The impulse response is $h(t) = \delta(t) + \delta(t-1) + \delta(t-2)$: notice that the non-zero values of $h(t)$ correspond to positive $t$'s. When we apply the convolution sum, we will need to be careful to take the reflection of $h$, so that $g(t)$ depends indeed on values of $f$ for $t' < t$.

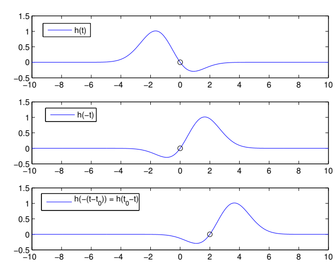


Figure 4: Reflection and shift of the impulse response when computing the convolution.

## *Fourier Transform*

Quick reminder on complex numbers:

- $a + jb$, $j^2 = -1$
- Harmonic exponentials $e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$.

$$\begin{aligned} f(t) \to & F(\omega) = \mathcal{F}\{f(t)\} \\ F :& \mathbb{R} \to \mathbb{C} \\ F(\omega) =& \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt, \end{aligned}$$

where $\omega$ denotes the frequency.

Inverse Fourier transform:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$$