CIS 580 Spring 2012 - Lecture 18

March 26, 2012

Notes and figures by Matthieu Lecce.

Review: examples of projective transformations

- There is a projective transformation between any horizontal plane $Z = h$ and the camera screen.
- More generally there is a projective transformation betwee any plane and the camera screen.
- A purely rotating camera induces a projective transformation.

Single-view geometry

Vanishing points

We consider the simple case where the intrinsic matrix of a camera takes the following form:

$$K = \begin{bmatrix} f & 0 & u_0 \\ 0 & f & v_0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Last time, we saw how to compute the focal length of a camera, given the projections of the vanishing points.

We saw that under the condition that vanishing points are not at infinity, we can derive f from the constraint on K.

Can we recover the image center from the projections of three orthogonal vanishing points?

Theorem: The image center (u_0, v_0) is the orthocenter of the triangle formed by the projections of three orthogonal vanishing points.

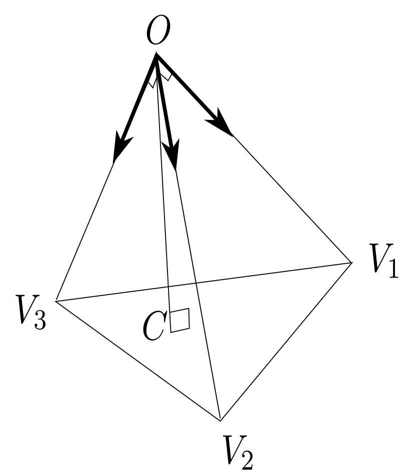

Figure 1: Three orthogonal vanishing points and image center.

We used the fact that the two non-parallel lines OV_2 and V_2V_3 contained in the plane OV_2V_3 are perpendicular to OCV_1, therefore $OV_2V_3 \perp OCV_1$.

Proof: See figure 1. Let $C = (u_0, v_0, 1)$ denote the homogeneous coordinates of the image center: it is defined as the intersection of image plane $V_1V_2V_3$ with the optical axis, which is the line through O and perpendicular to $V_1V_2V_3$.

$$\begin{aligned} OC \perp V_1V_2V_3 &\Rightarrow OCV_1 \perp V_1V_2V_3 \\ OV_1 \perp OV_2 &\Rightarrow OCV_1 \perp \text{any line contained in } V_1V_2V_3 \end{aligned}$$

In particular $OCV_1 \perp V_2V_3$. Moreover, $OV_1 \perp OV_2$, therefore $\boxed{OCV_1 \perp OV_2V_3}$.

When two planes are perpendicular, their intersections with a third plane are also perpendicular. Therefore, the intersection of OCV_1 with $V_1V_2V_3$ is perpendicular to intersection of OV_2V_3 with $V_1V_2V_3$.

In other words $V_1C \perp V_2V_3$. A similar reasoning leads to $V_2C \perp V_3V_1$ and $V_3C \perp V_1V_2$, therefore C is the orthocenter of $V_1V_2V_3$.

This elegant result would enable one to calibrate a camera using only the vanishing points in an image. In practice we don't use this method because it is reliable only when there is a strong perspective in the image, i.e. when vanishing points have low coordinates. When it is not the case, the intersection of parallel lines projected in the image have large coordinates (thousands of pixels), and the relative error is too big to guarantee precise calibration.

Cross-ratios

See fig. 3.

Definition Given four points A, B, C, D, we define the cross-ratio of their distances as follows: $CR(A, B, C, D) = \frac{AC}{AD} : \frac{BC}{BD}$

Invariance of the cross-ratio It is easy to prove that $CR(A, B, C, D)$ remains invariant under projective transformations $\mathbb{P}^n \to \mathbb{P}^n$:

$$\frac{AC}{AD} : \frac{BC}{BD} = \frac{A'C'}{A'D'} : \frac{B'C'}{B'D'}$$

Simple example Consider the following projective transformation in $\mathbb{P}^1$:

$$\begin{bmatrix} u' \\ w' \end{bmatrix} \sim \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} u \\ w \end{bmatrix}, \quad ad - bc \neq 0$$

We have the following:

$$\frac{u'}{w'} = \frac{au + bw}{cu + dw}$$

W.L.O.G we can set $w = w' = 1$, which yields:

$$u' = \frac{au + b}{cu + d}$$

If we consider four points of coordinates $u_1, u_2, u_3, u_4 \in \mathbb{R}^1$, we have:

$$\begin{aligned}
\frac{u'_3 - u'_1}{u'_4 - u'_1} : \frac{u'_3 - u'_2}{u'_4 - u'_2} &= \frac{\frac{au_3+b}{cu_3+d} - \frac{au_1+b}{cu_1+d}}{\frac{au_4+b}{cu_4+d} - \frac{au_1+b}{cu_1+d}} : \frac{\frac{au_3+b}{cu_3+d} - \frac{au_2+b}{cu_2+d}}{\frac{au_4+b}{cu_4+d} - \frac{au_2+b}{cu_2+d}} \\
&= \frac{(cu_4 + d)(cu_1 + d)}{(cu_3 + d)(cu_1 + d)} \frac{(cu_3 + d)(cu_2 + d)}{(cu_4 + d)(cu_2 + d)} \cdot \\
&\quad \frac{(au_3 + b)(cu_1 + d) - (au_1 + b)(cu_3 + d)}{(au_4 + b)(cu_1 + d) - (au_1 + b)(cu_4 + d)} \frac{(au_3 + b)(cu_2 + d) - (au_2 + b)(cu_3 + d)}{(au_4 + b)(cu_2 + d) - (au_2 + b)(cu_4 + d)} \\
&= \frac{(au_3 + b)(cu_1 + d) - (au_1 + b)(cu_3 + d)}{(au_4 + b)(cu_1 + d) - (au_1 + b)(cu_4 + d)} \frac{(au_3 + b)(cu_2 + d) - (au_2 + b)(cu_3 + d)}{(au_4 + b)(cu_2 + d) - (au_2 + b)(cu_4 + d)} \\
&= \frac{(ad - bc)(u_3 - u_1)}{(ad - bc)(u_4 - u_1)} \frac{(ad - bc)(u_3 - u_2)}{(ad - bc)(u_4 - u_2)} \\
&= \frac{u_3 - u_1}{u_4 - u_1} : \frac{u_3 - u_2}{u_4 - u_2}
\end{aligned}$$

Point ordering and cross-ratios For the same set of four points A, B, C, D, there are 24 ways to write a cross-ratio (permutations of A, B, C, D) to obtain λ,$1 - \lambda$,$\frac{1}{\lambda}$,$\frac{1}{1-\lambda}$.

Cross-ratios containing vanishing points The invariance holds even when the cross-ratio contains vanishing points. For $A, B, C \in \mathbb{P}^3$, we obtain the following result by starting from the cross-ratio defined above, and taking the limit when one of the points goes to infinity:

Notational abuse: $\infty \leftrightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$CR(A, B, C, \infty) = \lim_{D\to\infty} \frac{AC}{BC} : \frac{AD}{BD} = \frac{AC}{BC}$$

This trick provides us with a useful geometrical reasoning to "transfer" distances in an image:

- Let's assume we know the pixel positions of A', B', C', the projections of three points A, B, C,
- Let's assume we also know the pixel position of V', projection of the vanishing point V

We know that there is a projective transformation between the plane containing A, B, C and the camera screen. Therefore, if we know AC we can obtain BC and vice versa:

$$\underbrace{\frac{A'C'}{A'D'} : \frac{B'C'}{B'D'}}_{\text{Known pixel positions}} = \frac{AC}{BC}$$

EXAMPLE 1: See fig. 4. Given this picture, what is the distance to the finish line? *Assume D* is at infinity (the two lines are parallel):

$$\lambda_{px} = \frac{x}{1+x} \to x = \dots$$

.

EXAMPLE 2: Given a picture of the William Penn statue (Town Hall) and the Liberty tower # 1, and the horizon of the ground plane, find the height of the Lierty tower given the fact the W. Penn statue has a height from the ground of 167m.

See fig. 5.

The horizon is the intersection of two horizontal vanishing points. We know the vertical vanishing point of the line through the Liberty tower (LP=Liberty Place).

How can we find which point on Liberty place has a height of 167m? Q, the intersection of the horizon and AA', is a *horizontal vanishing point*, therefore any line through Q is parallel to AA' ! Therefore the point we are looking for is B (on figure 6, intersection of QB' and vertical line through LP).

In the world, the pre-image of BB'Q is parallel to the ground, which means that the pre-image of AB is 167m long.

$$\{A, B, L, V\}_{\text{pixels}} = \frac{AL}{A\infty} : \frac{BL}{B\infty} = \frac{AL}{BL} = \frac{AL}{AL-167} = f(AL)$$

Therefore we can transfer the length from Town Hall to Liberty place if we know the points above. In practive, A and A' are hard to find because of occlusions.

ANOTHER EXAMPLE: picture of a man standing in front of his house. See fig. 7.

$$\{A, B, C, V\}_{\text{pixels}} = \frac{AC}{A\infty} : \frac{BC}{B\infty} = \frac{AC}{BC} = \frac{AC}{AC - h} = \frac{220}{220 - h} \rightarrow h = \ldots$$