CIS 580 Spring 2012 - Lecture 2

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Notes and figures by Matthieu Lecce.

Last lecture's main result: linear shift-invariant (LSI) systems can be represented as a convolution.

The Fourier Transform

Quick reminder on complex numbers:

- $a + jb \in \mathbb{C}$, $j^2 = -1$
- $e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$.

Definition of the Fourier Transform:

$$
\begin{aligned}
f(t) \circ\!\!-\!\!\bullet F(\omega) &= \mathcal{F}\{f(t)\} \\
F: \mathbb{R} &\to \mathbb{C} \\
F(\omega) &= \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt,
\end{aligned}
$$

where ω denotes the frequency. This definition is sometimes called non-unitary Fourier transform, with angular frequency (ω is referred to as angular frequency, and s such that $\omega = 2\pi s$ is the ordinary frequency).

Inverse Fourier transform:

$$
f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega
$$

Domains:

- $f(t)$ defined in time (or space for x, y) domain
- $F(\omega)$ defined in frequency (or spatial frequency) domain

The Fourier transform can be defined as a function of s, the frequency, where $\omega = 2\pi s$, in which case the definitions can be rewritten as follows:

$$
\begin{aligned}
F(s) &= \int_{-\infty}^{\infty} f(t)e^{-j2\pi st}dt \\
f(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)e^{j2\pi st}ds
\end{aligned}
$$

Function symmetry and Fourier

Definitions:

- A function f_e is said *even* when $f_e(-t) = f_e(t)$
- A function f_o is said *odd* when $f_o(-t) = -f_o(t)$

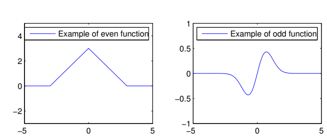

Figure 1: Even and odd functions.

Any function $f(t)$ can be decomposed into an odd and an even part:

$$
\begin{aligned}
f(t) &= f_e(t) + f_o(t) \\
\text{where } f_e(t) &= \frac{1}{2}(f(t) + f(-t)) \text{ is even} \\
f_o(t) &= \frac{1}{2}(f(t) - f(-t)) \text{ is odd}
\end{aligned}
$$

If we apply the Fourier transform to this decomposition, we obtain the following:

$$
\begin{aligned}
&\int_{-\infty}^{\infty}(f_e(t)+f_o(t))(\cos\omega t - jsin\omega t)dt \\
=&\int_{-\infty}^{\infty} f_e(t)\cos(\omega t)dt - j\int_{-\infty}^{\infty}(f_o\sin\omega t)dt
\end{aligned}
$$

The even part maps to the real part of the Fourier transform and the odd part to the imaginary part (and vice versa).

Observe that $\int_{-\infty}^{\infty} g_e(t)g_o(t)dt = 0$

Theorems

Shift theorem

$$f(t-t_0) \circ\!\!-\!\!\bullet F(\omega)e^{-j\omega t_0}$$

Example: for $f(t)$ even, $f(t-\frac{T}{2}) \circ\!\!-\!\!\bullet F(\omega)e^{-j\omega\frac{T}{2}}$, where $F(\omega)$ is real.

Modulation theorem

$$f(t)e^{j\omega_0 t} \circ\!\!-\!\!\bullet F(\omega-\omega_0)$$

Multiplying by a complex exponential causes a shift in the frequency domain.

Similarity theorem

$$f(at) \circ\!\!-\!\!\bullet \frac{1}{|a|}F(\frac{\omega}{a})$$

Convolution This theorem is extensively used in image processing:

$$
\begin{array}{ccccc}
f(t) & \rightarrow & \boxed{h(t)} & \rightarrow & g(t) = \int_{-\infty}^{\infty} g(t')h(t-t')dt' = f(t)*h(t) \\
\overset{\circ}{\underset{\bullet}{|}} & & \overset{\circ}{\underset{\bullet}{|}} & & \overset{\circ}{\underset{\bullet}{|}} \\
F(\omega) & & H(\omega) & & G(\omega)?
\end{array}
$$

What happens in the Fourier domain?

$$
\begin{aligned}
G(\omega) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t')h(t-t')dt'e^{-j\omega t}dt \\
&= \int_{t'=-\infty}^{\infty} f(t')\left\{\int_{t=-\infty}^{\infty} h(t-t')e^{-j\omega t}dt\right\}dt' \\
&= \int_{t'=-\infty}^{\infty} f(t')H(\omega)e^{-j\omega t'}dt' \text{ (shift theorem)} \\
&= H(\omega)F(\omega)
\end{aligned}
$$

$H(\omega)$ is call the transition function (as opposed to impulse response).

Inverse convolution

$$f(t)h(t) \circ\!\!-\!\!\bullet \frac{1}{2\pi}F(\omega)*H(\omega)$$

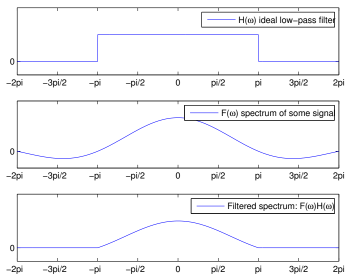

Figure 2: Low pass filtering by taking the product $H(\omega)F(\omega)$.

Fourier of some interesting functions

1. Recall the absorption property

$$\int_{-\infty}^{\infty} \delta(t-t_0) f(t) dt = f(t_0)$$

Then we have:

$$\int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1$$

We will remember the two following results:

$$\begin{aligned} \delta(t) \quad &\circ\!\!-\!\!\bullet \quad 1 \\ 1 \quad &\circ\!\!-\!\!\bullet \quad 2\pi\delta(t) \quad \text{(DC-component)} \end{aligned}$$

2. Fourier of a harmonic exponential $e^{j\omega_0 t}$:

$$\begin{aligned} e^{j\omega_0 t} \quad & \underbrace{\circ\!\!-\!\!\bullet}_{\text{modulation}} \quad 2\pi\delta(\omega-\omega_0) \\ \cos(\omega_0 t) \quad & \circ\!\!-\!\!\bullet \quad 2\pi.\tfrac{1}{2}(\delta(\omega-\omega_0)+\delta(\omega+\omega_0)) \\ \sin(\omega_0 t) \quad & \circ\!\!-\!\!\bullet \quad 2\pi.\tfrac{1}{2j}(\delta(\omega-\omega_0)-\delta(\omega+\omega_0)) \end{aligned}$$

Note: using the similarity theorem, we can derive simpler expressions using the ordinary frequency s instead of ω, for example:

$$\cos(2\pi s_0 t) \circ\!\!-\!\!\bullet \frac{1}{2}\delta(s+s_0) + \delta(s-s_0)$$

3. Fourier of the comb function $\text{III}(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT)$:

$$\sum_{n=-\infty}^{\infty} \delta(t-nT) \circ\!\!-\!\!\bullet \frac{1}{|T|} \sum_{n=-\infty}^{\infty} \delta(s-\frac{n}{T})$$

(or $\frac{2\pi}{|T|} \sum_{n=-\infty}^{\infty} \delta(\omega - \frac{2\pi n}{T})$ when using the non-unitary Fourier Transform with angular frequency).

4. Fourier of a 1D Gaussian The 1D Gaussian distribution is defined as follows:

$$f(t) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma^2}}$$

When trying to integrate an exponential that contains the variable to the power 2, we will always try to boil it down to the famous *Gaussian integral*:

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

Here is a loose proof of the Gaussian integral:

$$
\begin{aligned}
I &= \int_{-\infty}^{\infty} e^{-x^2} dx \\
I^2 &= \int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-y^2} dy \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx dy
\end{aligned}
$$

We perform a substitution to use polar coordinates, the integral I^2 now takes the following form:

$$
\begin{aligned}
x =& r\cos\theta \\
y =& r\sin\theta \\
dxdy =& rdrd\theta
\end{aligned}
$$

$$
\begin{aligned}
I^2 =& \int_{r=0}^{\infty} \int_{\theta=0}^{2\pi} e^{-r^2} rdrd\theta \\
=& 2\pi \int_{r=0}^{\infty} re^{-r^2} dr \\
=& 2\pi \left(\frac{1}{-2}\right) \left[e^{-r^2}\right]_0^{\infty} \qquad ((e^{-r^2})' = -2re^{-r^2}) \\
=& \pi
\end{aligned}
$$

NOW LET'S COMPUTE the Fourier transform of a Gaussian distribution:

$$
\begin{aligned}
\mathcal{F}\{f(t)\} =& \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{t^2}{2\sigma^2}} e^{-2\pi st} dt \\
=& \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{t^2}{2\sigma^2} - 2\pi s \frac{t}{\sigma\sqrt{2}} \sigma\sqrt{2} - \pi^2 s^2 \delta^2 2 + \pi^2 s^2 2} dt \\
=& \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(t + j\pi\sigma s\sqrt{2})^2} \\
=& \frac{1}{\sigma\sqrt{2\pi}} e^{-2\pi^2\sigma^2 s^2} \sqrt{2}\pi
\end{aligned}
$$

Therefore the Fourier Transform of a Gaussian is a Gaussian in terms of s!

Sampling

Definition and problem

Sampling is a multiplication of the signal by the comb function $Ш_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$. T is the sampling interval:

$$
f_S(t) = f(t) \sum_{n=-\infty}^{\infty} \delta(t - nT)
$$

(S for "sampled") After sampling, we forget about the T: we just obtain a sequence of numbers, that we note $f_s[k]$.

Examples:

(a) Video of a rotating wheel. Depending on T, no motion is perceived, or even worse, a backward motion is perceived. (We will cover this example in HW2)

(b) Sinusoidal signal: appearant frequency of sampled signal is different

(c) Same for a checker pattern (Sequence of step functions)

Problem: we want to find the lowest possible sampling frequency, such that the sampled signal is not corrupted. **Solution**: Let's use the frequency domain to analyze the action of the comb (sampling) function.

Sampling in the frequency domain

Let's compute the Fourier transform of the comb function:

$$\sum_{n=-\infty}^{\infty} \delta(t - nT) \circ\!\!-\!\!\bullet \sum_{n=-\infty}^{\infty} \delta(\omega - \frac{2\pi n}{T}) = \sum_{n=-\infty}^{\infty} \delta(s - \frac{n}{T})$$

Remember we defined the frequency s where $\omega = 2\pi s$

Sampling in the time domain is a **multiplication** with the comb function $\text{III}(t)$, therefore in the frequency domain it is a **convolution** with the Fourier of the comb, which is a sum of impulses $\delta(\omega - \omega_0)$.

The convolution with one impulse $\delta(\omega - \omega_0)$ corresponds to shifting the spectrum such that it is centered around w_0 instead of 0.

While in the time domain sampling is very simple, in the Fourier domain it is a complete mess: it is equivalent to "xeroxing" the signal (making several shifted copies of it) in the frequency domain:

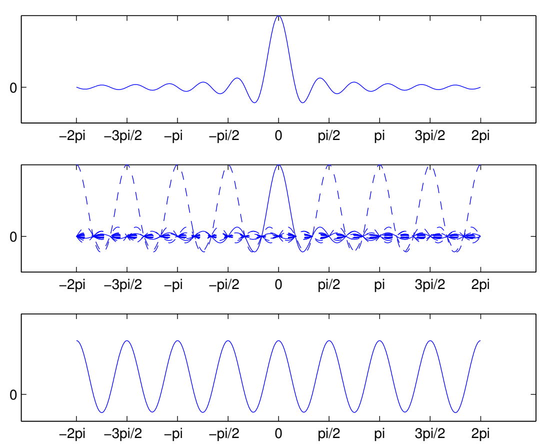

Figure 3: Sampling in the time domain corresponds to xerozing in the Fourier domain. Plot 1: spectrum of a rectangle (sinc function). Plot 2: spectrum and replicas after convolving with impulses. Plot3: result of convolution with comb function (sum of spectrum and replicas).

Fourier transform of a discrete signal To recover the signal, we need to be able to *isolate* the spectrum from its replicas: generally this is done by applying a low-pass filter $\text{rect}(T_s)$(like in figure 2). Even if the signal is band-limited, i.e. with maximum frequency ω_{max}, we need to have:

$$\underbrace{\omega_{\text{sampling}}}_{=\frac{2\pi}{T_s}} \geq 2\omega_{\text{max}}, \quad \text{i.e.} \quad \omega_{\text{max}} \leq \frac{\pi}{T_s}$$

This result is known as the **sampling theorem**.