CIS 580 Spring 2012 - Lecture 23

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Reconstruction from two calibrated views

Structure from motion, epipolar geometry

We want to estimate the transformation between two camera and the 3D positions of a set of observed points, given 2D-2D correspondences in the two images: (u_1, v_1) matched to (u_2, v_2).

Assuming a simple camera model with focal length f, the calibrated rays are directed by the following vectors:

$$p_1 = \begin{bmatrix} \frac{u_1 - c_{x1}}{f_1} \\ \frac{v_1 - c_{y1}}{f_1} \\ 1 \end{bmatrix}, \quad p_2 = \begin{bmatrix} \frac{u_2 - c_{x2}}{f_2} \\ \frac{v_2 - c_{y2}}{f_2} \\ 1 \end{bmatrix}$$

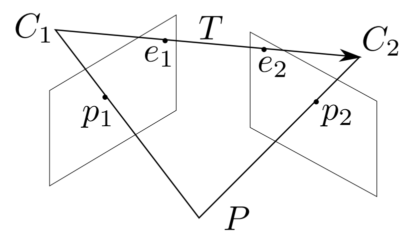

Figure 1: R, T transform the axes of 1 into the axes of 2, therefore in terms of coordinates $P_1 = RP_2 + T$.

The epipolar constraint relates p_1, p_2 and the motion parameters:

$$\boxed{p_1^T (T \times R p_2) = 0}, \tag{1}$$

and it just states that the two calibrated rays are coplanar in a plane that goes through the baseline T.

Note that $\lambda_2 p_2 = R\lambda_2 p_2 + T$, but the epipolar constraints enables us to get rid of the depth.

If we fix p_1, the constraint defines a line which is simply the projection of the ray p_1 in the screen of 2. Such a line is called an **epipolar line**.

All epipolar lines in 2 go through e_2, the projection of C_1 in the screen of 2. e_1, e_2 are called the **epipoles**.

Geometric interpretation: In figure 1, notice that the plane C_1C_2P intersects screen 1 at the epipolar line $e_1 \times p_1$, and it intersects screen 2 at the epipolar line $e_2 \times p_2$. Both lines are defined from the epipolar constraint (eq. 1), by fixing p_1 or p_2. Using this geometric interpretation, it is easy to derive e_1, e_2 either from the epipolar constraint:

$$e_1 \sim T, \quad e_2 \sim -R^T T$$

Note on coordinate transforms If the axes of camera 2 are translated of T and rotated of R with respect to the axes of camera 1, the relationship between the coordinates is the following:

$$P_1 = RP_2 + T$$

Notice that what happens to the coordinates is the "opposite" of what happens to the axes: if C_2 is translated of 100 along X with respect to C_1, then we need to *subtract* 100 to the X-coordinate of P_1 to obtain P_2 if P_1, P_2 describe the same point P

Therefore we have another way to derive e_1, e_2:

$$\begin{aligned} e_1 &= R.0 + T, \quad \text{i.e. } e_1 = T \\ 0 &= Re_2 + T, \quad \text{i.e. } e_2 = -R^T T \end{aligned}$$

The essential matrix

We define the essential matrix $\boxed{E = \widehat{T}R} \in \mathbb{R}^{3\times3}$, where $\widehat{T} = \begin{bmatrix} 0 & -t_z & -t_y \\ t_z & 0 & -t_x \\ -t_y & t_x & 0 \end{bmatrix}$

Using the anti-symmetric matrix $\widehat{T}$ enables us to write the cross-product with T as a matrix-vector product: for any $P \in \mathbb{R}^3$, $T \times P = \widehat{T}P$.

The epipolar constraint can then be reformulated very simply:

$$p_1^T E p_2 = 0$$

Naturally, this linear equation involving the parameters of E encourage us to estimate E using several pairs of matching points (p_1, p_2). The interest of estimating the essential matrix is two-fold:

- Once we have E, we can plot epipolar lines: given a point p_2, then the corresponding point in the first image is on the following line:

$$\begin{bmatrix} x & y & 1 \end{bmatrix} \underbrace{Ep_2}_{3\times1} = 0$$

- Since E is defined with the help of T and R, we can first estimate E from a set of correspondences, and then recover the motion parameters from E.

Estimating E As usual, we want to formulate and solve a linear system $Ae = 0$ where A is some matrix containing the point positions and e contains the parameters of E, to estimate. Given a pair of matching points p_1, p_2, we can further rewrite the constraint as follows, writing E in form of a vector:

$$\underbrace{\begin{bmatrix} p_{2x}p_1^T & p_{2y}p_1^T & p_{2z}p_1^T \end{bmatrix}}_{1\times9} \underbrace{\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}}_{9\times1} = 0 \qquad (2)$$

The vector $\begin{bmatrix} p_{2x}p_1 \\ p_{2y}p_1 \\ p_{2z}p_1 \end{bmatrix} \in \mathbb{R}^9$ is called the Hadamard product of p_1 and p_2 and is noted $p_1 \otimes p_2$.

Therefore equation 2 can be rewritten as $(p_1 \otimes p_2)^T e_s = 0$.

where $e_s = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} \in \mathbb{R}^9$ is just E written as a vector, i.e. $E = \begin{bmatrix} | & | & | \\ e_1 & e_2 & e_3 \\ | & | & | \end{bmatrix}$.

For every correspondence p_1, p_2 we obtain a linear homogeneous equation w.r.t. $\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = e_S$: we need **8 independent equations**, i.e. 8 matching points.

E can be estimated up to a scale factor

The 8-point algorithm

By stacking eight independent equations as defined in equation 2, we obtain a system of form $Ae_S = 0$ (A is 8×9): e_s is in the **null-space** of A, hence $e_s = v_9$ if $A = U\Sigma V^T$ is the SVD of A and $V = \begin{bmatrix} | & | & | & | \\ v1 & v_2 & \dots & v_9 \\ | & | & | & | \end{bmatrix}$.

If we recover E with this method, are we sure it will be an "acceptable" essential matrix? We have to define what it means from a matrix to be essential: concretely it means that is is the product of an antisymmetric and a special orthogonal matrix. Can any 3×3 real matrix E be decomposed into $E = \widehat{T}R$?

We know other examples of decompositions that work for *any* 3×3 matrix:

- QR factorization: $A = QR$ (Q orthogonal, R upper triangular),
- SVD decompositions, $A = U\Sigma V^T$

Is "$\widehat{T}R$" a valid decomposition for any 3×3 matrix?

E-properties

Here are a few properties that will be useful in subsequent derivations:

- $E^T = R^T\widehat{T}^T$
- $\widehat{T}^T = -\widehat{T}, \widehat{T}a = T \times a$
- $E^T T = 0$.
- $\det(E) = \det\widehat{T}.\det R = 0$, therefore $\sigma_3 = 0$ (σ_3 is the smallest singular value of E)

Remember that the determinant of a matrix is the product of its eigenvalues, and for any matrix A, the null-space of A is the same as the null-space of $A^T A$.

IN THIS SECTION, we are going to show that *an essential matrix can be characterized by its singular values.* Namely if E is an essential matrix (i.e. decomposable into $\widehat{T}R$), its singular values verify $\sigma_1 = \sigma_2$, and $\sigma_3 = 0$. We just proved that $\sigma_3 = 0$.

Remember that the singular values of E are the eigenvalues of $E^T E$ or EE^T, i.e. the solutions of the following characteristic polynomial:

$$\det EE^T - \sigma I = 0.$$

We want to characterize the singular values without solving the characteristic polynomial. We exploit the fact that $R^T R = I$ to find a nice form for EE^T:

$$\begin{aligned} EE^T &= \widehat{T}\widehat{T}^T \\ &= TT^T - T^T T I \\ &= \begin{bmatrix} t_x^2 & t_x t_y & t_x t_z \\ 0 & t_y^2 & t_y t_z \\ 0 & 0 & t_z^2 \end{bmatrix} - \|T\|^2 I \end{aligned}$$

If $\|T\| = 1$, then there exists a rotation U such that $UT = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = T_z$

We can always rotate a vector to align it to the Z-axis.

And we are going to try to express EE^T as a function of the following simple matrix:

$$\widehat{T_z}\widehat{T_z}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

LEMMA: If Q is orthogonal ($Q^T Q = I$), then

$$\widehat{Qa} = Q\widehat{a}Q^T$$

PROOF: $\widehat{Qa}b = Qa \times b = Q(a \times Q^T b) = Q\widehat{a}Q^T b$.

We are now able to express $\widehat{T}$ as $\widehat{T_z U^T} = U^T \widehat{T} U$

$$\begin{aligned} EE^T = \widehat{T}\widehat{T}^T &= U^T \widehat{T}_z U U^T \widehat{T}_z^T U \\ &= U^T \widehat{T}_z \widehat{T}_z^T U \\ &= U^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} U \end{aligned}$$

EE^T is symmetric hence $\sigma_1 = 1, \sigma_2 = 1, \sigma_3 = 0$. We can now formulate conditions that characterize essential matrices.

NECESSARY CONDITION: If E is an essential matrix (i.e. $E = \widehat{T}R$) then E has two equal singular values and one singular value equals zero.

SUFFICIENT CONDITION:

$$\begin{aligned} E &= U \begin{bmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & 0 \end{bmatrix} V^T \\ &= \sigma U \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} V^T \\ &= \sigma U \widehat{T}_z^T R_z V^T \\ &= \sigma \underbrace{\widehat{UT_z}}_{\text{antisymmetric}} \underbrace{URV^T}_{\text{orthogonal}} \end{aligned}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = T_z^T R_{z,\pi/2}$$

Observe $UT_z = U \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, which is the last column of U

We just showed that there is at least one such decomposition, but is it unique?

NECESSARY AND SUFFICIENT CONDITION: E is essential **iff** $\sigma_1(E) = \sigma_2(E) \neq 0$ and $\sigma_3(E) = 0$.

How many ways can we decompose E?.

This is important, not just for mathematical purposes: we need to find *all* solutions of the system, in case some of them are not correct (do not satisfy some constraints)

We showed the following decomposition:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{-\widehat{T}_z} \underbrace{\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{R_{z,\pi/2}}$$

But we could similarly write $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \widehat{T}_z R_{z,-\pi/2}$.

Pose recovery from the essential matrix

If $E = U\Sigma V^T = U\begin{bmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & 0 \end{bmatrix}V^T$, there are two solutions for the pair $(\widehat{T}, R)$:

$$
\begin{aligned}
(\widehat{T_1}, R_1) &= (UR_{z,+\pi/2}\Sigma U^T, UR^T_{z,+\pi/2}V^T) \\
(\widehat{T_2}, R_2) &= (UR_{z,-\pi/2}\Sigma U^T, UR^T_{z,-\pi/2}V^T)
\end{aligned}
$$