CIS 580 Spring 2012 - Lecture 3

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Notes and figures by Matthieu Lecce.

Review from last lecture:

- Sampling (or time-sampling) is a multiplication with the comb function $\text{Ш}(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$ (infinite trail of Dirac functions).
- It corresponds to a *convolution* with $\sum_{n=-\infty}^{\infty} \delta(\omega - \frac{2\pi n}{T})$ in the frequency domain.
- Concretely this means that in the frequency domain, the signal is *replicated* ("xeroxed") at frequencies $\frac{2\pi n}{T},\ n \in \mathbb{Z}$
- **Question**: Can we recover the original signal? In other words, can we isolate the original Fourier of $f(t)$ for this convolution (replication).
- **Answer**: Yes, *if* $2\omega_{\text{max}} \leq \omega_{\text{sampling}} = \frac{2\pi}{T}$, in which case we multiply the Fourier of the sampled function with a rectangle function (low-pass filter).

Fourier and sampling

Reconstructing a sampled signal

$$F_S(\omega) = F(\omega) * \sum_{n=-\infty}^{\infty} \delta(\omega - \frac{2\pi n}{T})$$

THE RECTANGLE FUNCTION $\Pi(t)$ is defined as follows:

$$\Pi(t) = \begin{cases} \frac{1}{2\pi} & -\pi \leq \omega \leq \pi \\ 0 & \text{anywhere else} \end{cases}$$

What signal do we recover by multiplying with $\Pi(\omega)$?

$$f(t) \sum_{n=-\infty}^{\infty} \delta(t - nT) * \mathcal{F}^{-1}(\Pi(\omega))$$

Let's compute the Fourier transform of a box (rect) filter:

$$\Pi(t) = \text{rect}(t) \begin{cases} 1 & |t| \leq \frac{1}{2} \\ 0 & \text{anywhere else} \end{cases}$$

Similarly, the inverse transform is the following (this is the one we need to understand the effect of low-pass filtering)

$$\begin{aligned}
\mathcal{F}\{\text{rect}(t)\} &= \int_{-1/2}^{1/2} 1 e^{-j\omega t} dt \\
&= \frac{1}{-j\omega} \left[e^{-j\omega t} \right]_{-1/2}^{1/2} \\
&= \frac{1}{-j\omega} \left[e^{-j\omega/2} - e^{j\omega/2} \right] \\
&= \frac{1}{-j\omega} \left(-2j \sin \frac{\omega}{2} \right) \\
&= \frac{\sin \frac{\omega}{2}}{\frac{\omega}{2}} = \text{sinc}\left(\frac{\omega}{2}\right)
\end{aligned}$$

$$
\begin{aligned}
\mathcal{F}^{-1}\{\text{rect}(\omega)\} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \text{rect}(\omega) e^{j\omega t} dt \\
&= \frac{1}{2\pi}\int_{-1/2}^{1/2} e^{j\omega t} dt \\
&= \frac{1}{2\pi}\frac{1}{j\omega}\left[e^{j\omega t}\right]_{-1/2}^{1/2} \\
&= \frac{1}{2\pi}\text{sinc}\left(\frac{t}{2}\right)
\end{aligned}
$$

We will remember the following definititions and results:

$$
\begin{aligned}
\frac{1}{2\pi}\text{sinc}(t/2) &\circ\!\!-\!\!\bullet \text{rect}(\omega) \\
2\pi\text{sinc}(\pi t) &\circ\!\!-\!\!\bullet \frac{1}{2\pi}\text{rect}\left(\frac{\omega}{2\pi}\right)
\end{aligned}
$$

Definitions:

$$
\Pi(\omega) = \begin{cases} \frac{1}{2\pi} & |\omega| \leq \pi \\ 0 & \text{anywhere else} \end{cases}
$$

$$
\text{rect}(\omega) = \begin{cases} \frac{1}{2\pi} & |\omega| \leq \frac{1}{2} \\ 0 & \text{anywhere else} \end{cases}
$$

Remember the scaling theorem:

$$
f(\alpha t) \circ\!\!-\!\!\bullet \frac{1}{|\alpha|} F\left(\frac{\omega}{t}\right)
$$

Reconstructed signal:

$$
\begin{aligned}
f_{\text{reconstr}}(t) &= f(t)\sum_{n=-\infty}^{\infty} \delta(t-nT) * 2\pi\text{sinc}(\pi t) \\
&= \sum_{n=-\infty}^{\infty} f[n] 2\pi\text{sinc}(\pi t)
\end{aligned}
$$

Discrete Fourier Transform

Definition of the Discrete Fourier Transform, for a discrete signal $f[n]$:

$$
f[n] \circ\!\!-\!\!\bullet \sum_{n=0}^{L-1} f[n] e^{-j\frac{2\pi k}{L}n} = F[k]
$$

Note the implicit definition $\omega = \frac{2\pi k}{L}$

with finite length: $n = 0 \ldots L-1$

Important: A discrete signal still has a continuous Fourier transform! The Discrete Fourier Transform corresponds to a *sampling in the (continuous) frequency domain.*

$$
f[n] \circ\!\!-\!\!\bullet \sum_{n=0}^{L-1} f[n] e^{-j\omega n}
$$

What does sampling in the frequency domain (as in figure 1) correspond to in the time/space domain? It corresponds to a **convolution**:

- FREQUENCY DOMAIN: multiplication with $\sum \delta(\omega - \frac{2\pi k}{L})$
- TIME DOMAIN: convolution with with $\sum \delta[n - kL]$, equivalent to **replication of the signal at multiples of its length**.

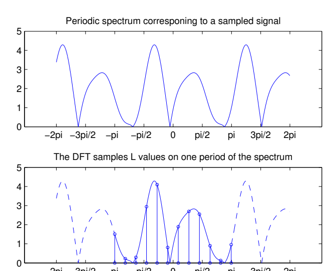

Figure 1: The DFT samples L values of one period of the spectrum by multiplying by the comb $\sum \delta\left(\omega - \frac{2\pi k}{L}\right)$

Fourier and derivation

Exact derivative in the Fourier domain

We are interested in the Fourier of the *derivative* of a function:

$$\begin{aligned} f(t) &\circ\!\!-\!\!\bullet F(\omega) \\ \frac{df(t)}{dt} &\circ\!\!-\!\!\bullet ? \end{aligned}$$

Bad idea:

$$\int_{-\infty}^{\infty} \frac{df}{dt} e^{-j\omega t} dt$$

Smart idea:

$$\begin{aligned} f(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega \\ \frac{df(t)}{dt} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) \boxed{j\omega} e^{j\omega t} \\ &= j\omega \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{j\omega t} d\omega \\ &= j\omega F(\omega) \end{aligned}$$

Therefore, when taking a **DFT** (spectrum is periodic), the derivation is equivalent to multiplying the spectrum by the periodic function represented by figure 2:

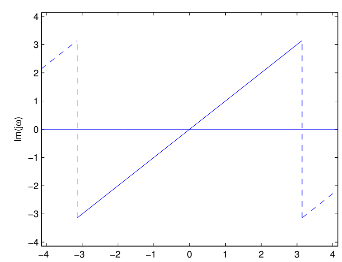

Figure 2: Derivating is equivalent to multiplying the spectrum by $j\omega$

Approximating the derivation with a filter

In many applications we need the derivative of the signal with respect to x or t. The goal of this section is to approximate the function in figure 2 (multiplication by $j\omega$ in the frequency domain) with an LSI filter (we will call it a derivative filter). More specifically, we want to find a discrete filter of impulse response $h[k],\ k = 1...K$ (K small) such that:

1. The DFT of h is close to $j\omega$
2. We "dump" high frequencies (DFT also corresponds to box sampling in frequency domain, so no approximation is needed for high frequencies)

Then, when we are given a discrete signal $f[n]$ represented in figure 3, all we have to do is to apply the filter to obtain $g[n]$, an approximation of the derivative of f.

$$g[n] = \sum_{l=-\infty}^{\infty} f[l] h[n-l],$$

Preliminary: The step function is defined such that $\frac{d}{dt}u(t) = \delta(t)$

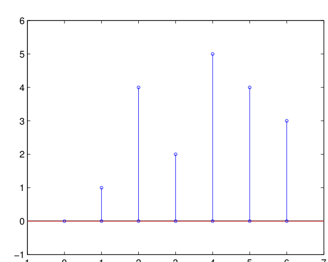

Figure 3: Discrete signal f[n]: how to compute an approximation of the derivative of f[n]?

The sign function comes handy: $\text{sgn}(t) = 2u(t) - 1, \; u(0) = 1/2$, or equivalently $u(t) = \frac{1}{2} + \frac{1}{2}\text{sgn}(t)$.

We have the following straightforward Fourier transforms:

$$u(t) \circ\!\!-\!\!\bullet \mathcal{U}(\omega) = \frac{1}{2}\delta(\omega) + \frac{1}{j\omega}$$
$$\frac{d}{dt}\text{sgn}(t) = 2\delta(t) \circ\!\!-\!\!\bullet j\omega\mathcal{F}(\text{sgn}(t)) = 2.$$

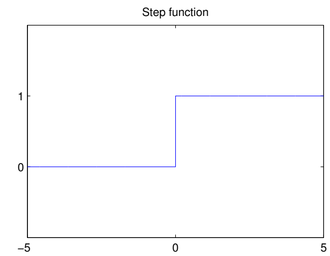

Figure 4: Step function.

A simple derivative filter Let's take the example of the following simple filter designed to approximate the derivative:

$$\begin{aligned} g[n] &= f[n] - f[n-1] \\ h[l] = \{1, -1\} \circ\!\!-\!\!\bullet H(\omega) &= \sum_{l=0}^{1} h[l] e^{-j\omega l} \\ &= 1 - e^{-j\omega} \\ &= 1 - \cos\omega + j\sin\omega \\ &= 2\sin^2\frac{\omega}{2} + j2\sin\frac{\omega}{2}\cos\frac{\omega}{2} \\ &= 2\sin\frac{\omega}{2}(\cos\frac{\omega}{2} + j\sin\frac{\omega}{2}) \end{aligned}$$

Therefore the magnitude of the transform is the following

$$|H(\omega)| = \left|2\sin\frac{\omega}{2}\right|$$

(to be continued)