CIS 580 Spring 2012 - Lecture 5

February 2, 2012

Notes and figures by Matthieu Lecce.

2D Fourier Transform

Review:

$$f(x,y) \circ\!\!-\!\!\bullet \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j\omega_x x+\omega_y y}dxdy$$
$$F(\omega_x,\omega_y) \bullet\!\!-\!\!\circ \left(\frac{1}{2\pi}\right)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\omega_x,\omega_y)e^{\cdots}d\omega_x d\omega_y$$

RESULTS FROM LAST LECTURE:

- Shift
- Affine transformation:

$$f\left(A\left[\begin{array}{c} x \\ y \end{array}\right]\right) \overset{2D}{\circ\!\!-\!\!\bullet} \frac{1}{det(A)}F\left(A^{-T}\left[\begin{array}{c} \omega_x \\ w_y \end{array}\right]\right))$$

- *Separability*: if a function is separable, the Fourier transform is separable too and the Fourier transform can be applied separately as a 1D transform along each axis.

  Examples:

- $f(x,y) = \cos(\omega_0 x) \circ\!\!-\!\!\bullet \frac{1}{2}(\delta(\omega_x - \omega_0) + \delta(\omega_x + \omega_0)).\delta(\omega_y)$
- $f(x,y) = \cos(\omega_1 x)\cos(\omega_2 y)$ corresponds to four points in the Fourier domain.
- $\cos(\frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y)$: two points in the Fourier domain, along the direction $\pi/4$

Sampling in 2D

The 2D equivalent of the comb function is a "bed of nails":

$$\sum_n\sum_m \delta(x-n\Delta x, y-m\Delta y) = \sum_n \delta(x-n\Delta x)\sum_m \delta(y-m\Delta y) \circ\!\!-\!\!\bullet \sum_n\sum_m \delta\left(s_x - \frac{n}{\Delta x}, s_y - \frac{m}{\Delta y}\right)$$

In order to avoid sampling artifacts, we need a result similar to the 1D case:

$$\Delta x \leq \frac{\lambda_{x\,\min}}{2}$$
$$\Delta y \leq \frac{\lambda_{y\,\min}}{2}$$

Rotation

Using the result on affine transformations, we have:

$$f\left(R\begin{bmatrix} x \\ y \end{bmatrix}\right) \circ\!\!-\!\!\bullet\; F\left(R\begin{bmatrix} \omega_x \\ \omega_y \end{bmatrix}\right)$$

Therefore the spectrum of $\cos(\omega_0(x\cos\phi - y\sin\phi))$ corresponds to two points along the direction ϕ, at distance ω_0 of the origin.

Wave signals

Consider an xyt-signal (video), in which we keep only one line (fixed y) of each frame. The effect of a translation in the video can be seen as a *wave* moving along the line at fixed y at constant speed u (in px/frame):

$$f(x,t) = f_0(x - ut)$$

Example: $f_0(x) = \cos(\omega_0 x)$, and $f(x,t) = \cos(\omega_0(x - ut))$

The Fourier transform of f is the following:

$$F(\omega_x, \omega_t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_0(x - ut)e^{-j\omega_x x + \omega_t t}dxdt$$

If we note $f_0(x) \circ\!\!-\!\!\bullet F_0(\omega_x)$ and make the substitution $x' = x - ut$, we have

$$\begin{aligned} F(\omega_x, \omega_t) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_0(x')e^{-j\omega_x(x'+ut)+\omega_t t}dx'dt \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_0(x')e^{-j\omega_x x'}e^{-j(\omega_x u + \omega_t)t}dx'dt \\ &= F_0(\omega_x)\delta(\omega_x u + \omega_t) \end{aligned}$$

Representing the spectrum $F(\omega_x, \omega_t)$ of the wave is equivalent to taking the original $F_0(\omega_x)$ spectrum, rotating it and stretching it by $\sqrt{1+u^2}$.

Examples:

- In our initial example $F_0(\omega_x) = \frac{1}{2}(\delta(\omega_x + \omega_0) + \delta(\omega_x - \omega_0))$: the $F_0(\omega_x, \omega_t)$ is made of two diracs located at $(\omega_0, -u\omega_0)$ and $(-\omega_0, +u\omega_0)$, which are at distance $\omega_0\sqrt{1+u^2}$ of the origin.
- If the spectrum $F_0(\omega_x)$ is a Gaussian, we have:

$$f_0(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{x^2}{2\sigma^2}} \circ\!\!-\!\!\bullet\; F_0(\omega_x) = e^{-\sigma^2\omega_x^2/2},$$

which again corresponds to "rotating and stretching" the Gaussian.

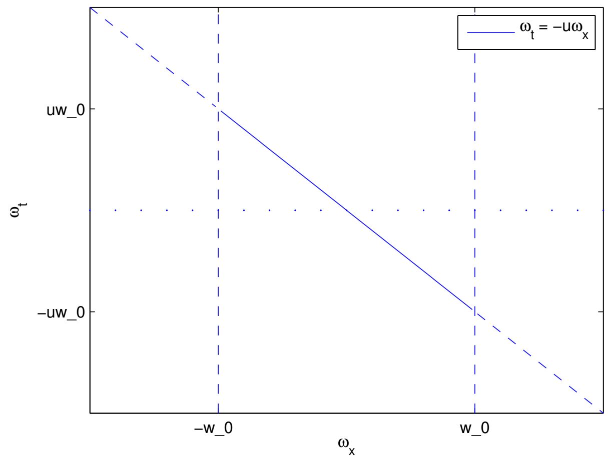

Figure 1: Wave signal in the xt Fourier domain: the spectrum $F_0(\omega_x)$ is multiplied by the indicator of te line $\omega_t = -u\omega_x$

Aliasing in video (Simoncelli, 1991)

Sampling in time is a multiplication by $\sum_{n=-\infty}^{\infty} \delta(t - n\Delta t)$. It corresponds to a convolution with $\sum_{n=-\infty}^{\infty} \delta(s_t - \frac{n}{\Delta t})$ (or $\sum_{n=-\infty}^{\infty} \delta(\omega_t = \frac{2\pi n}{\Delta t})$).

See 7. Reconstruction by boxing: aliasing causes perception of a speed of opposite sign.

$$\underbrace{u\omega_0}_{\max \omega_t} \leq \frac{\pi}{\Delta t} = \frac{\omega_{\text{sampling}}}{2}$$

Filters for detection

How to build a filter to recognize a cosine wave $\cos(\omega_0 x)$? In the frequency domain it corresponds to two impulses.

It is not possible to build a filter that looks like an impulse in Fourier domain, because it would correspond to a function of infinite support in time domain. Instead we could use a box filter: $H(\omega) = \Pi_{a/2}(\omega_0)$. The inverse function is still complicated $(\text{sinc}(\pi x/q)e^{j\omega_0 x}$, the harmonic exponential is here for modulation). A better candidate is a Gaussian filter centered around the frequency to detect, since the inverse transform of a Gaussian is a Gaussian.

(to be continued)