Machine Perception
CIS 580

Motion Estimation

Kosta Derpanis

February 22, 2012

UNIVERSITY *of* PENNSYLVANIA

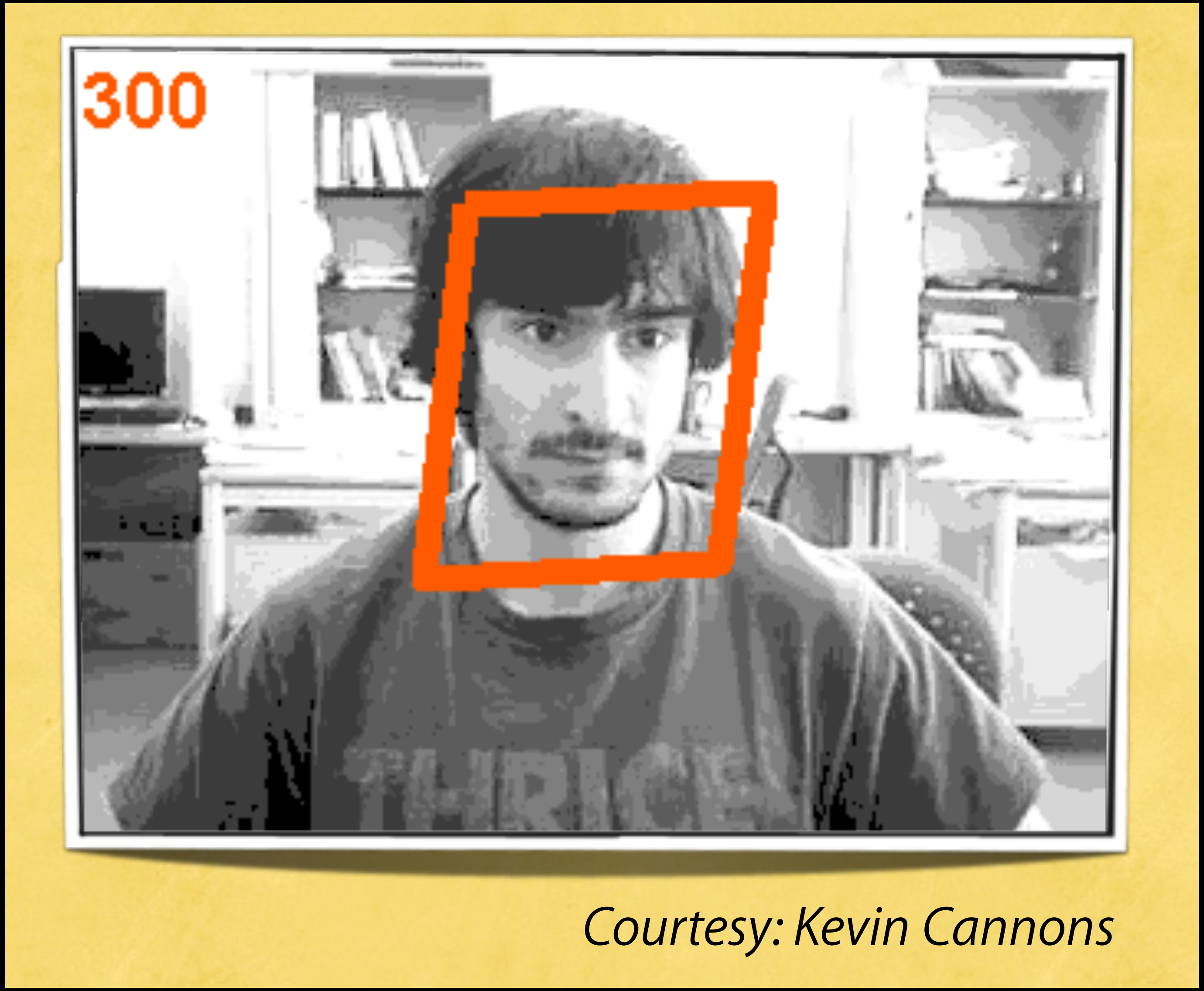
Announcements

Additional resources:

www.cvr.yorku.ca/members/gradstudents/kosta/compvis/

Why estimate motion?

Track objects



Courtesy: Kevin Cannons

Why estimate motion?

Track objects

Courtesy: Boris Babenko

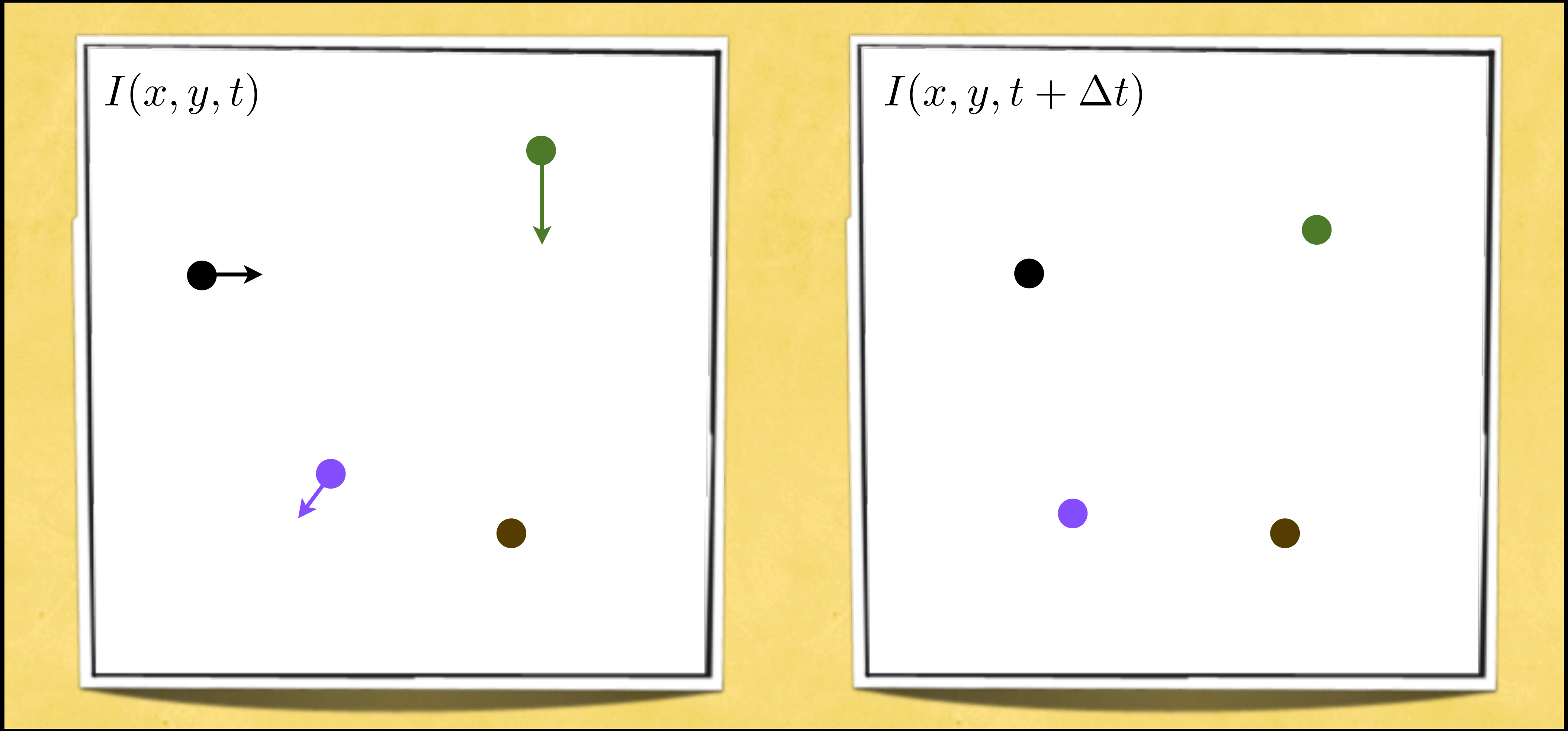
Why estimate motion?

Video stabilization

Why estimate motion?

Understanding behavior

Problem definition: Motion estimation



Estimate pixel motion between successive images

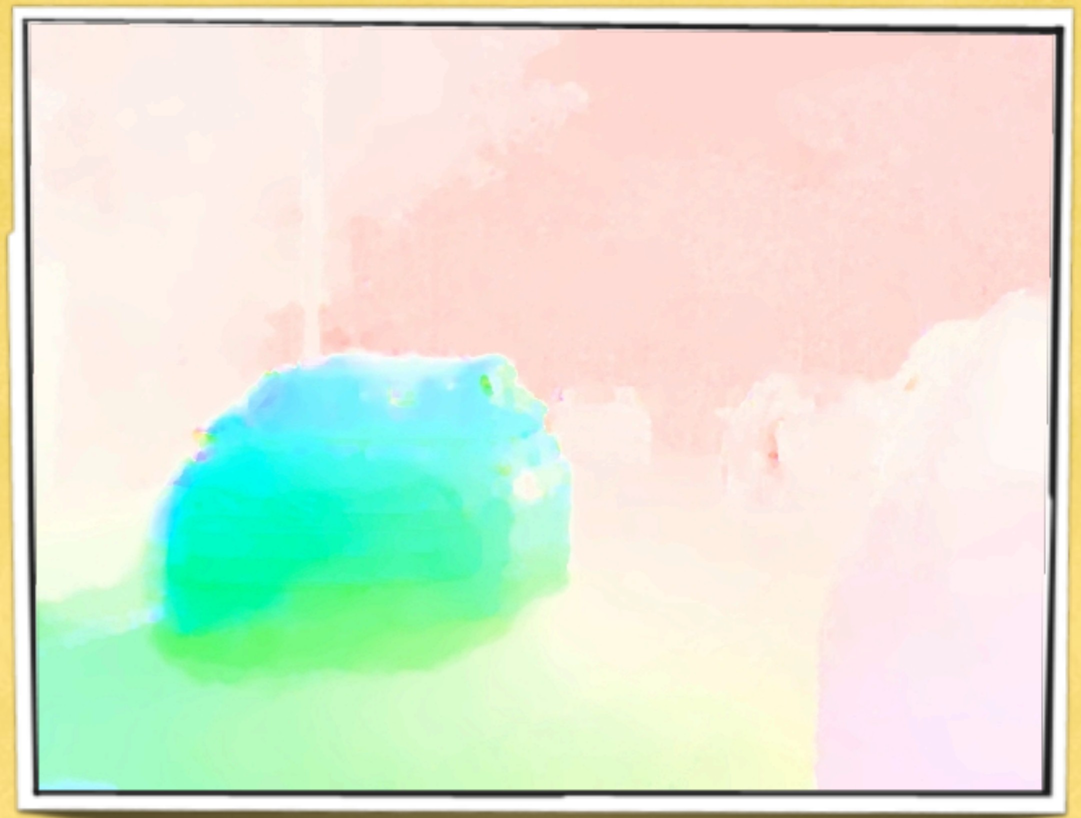
Key assumption: Brightness remains constant

Problem definition: Motion estimation

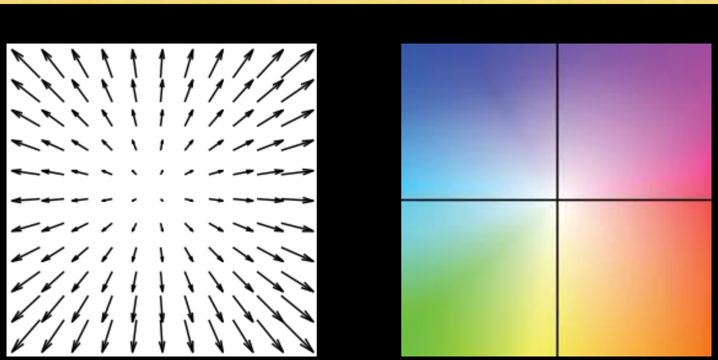
input sequence

Problem definition: Motion estimation

input sequence



optical flow estimate



Problem definition: Motion estimation

input sequence

stabilized sequence

Beyond scope

Beyond scope

Beyond scope

Beyond scope

Today’s agenda

Motion Field
vs.
Optical Flow

Today’s agenda

Motion Field
vs.
Optical Flow

Derivation
of
Brightness Constancy
Constraint

Today's agenda

Motion Field
vs.
Optical Flow

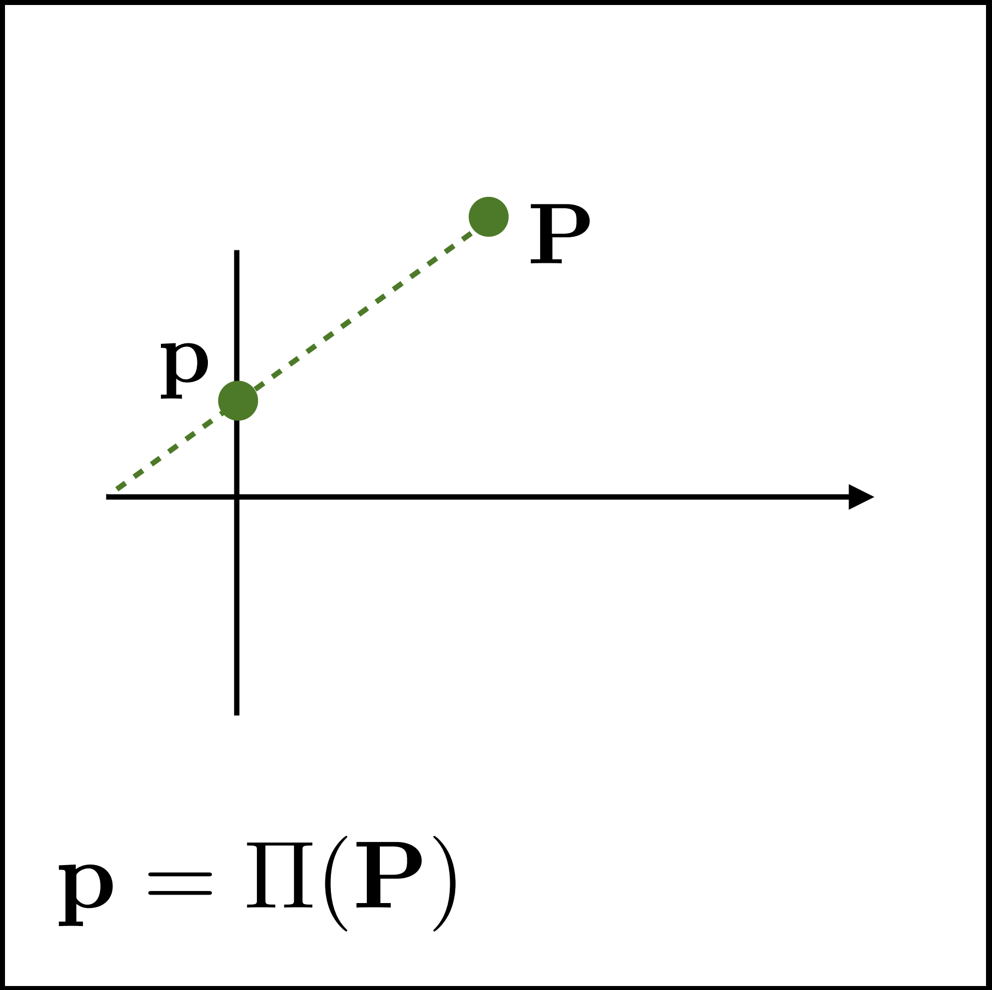
Derivation
of
Brightness Constancy
Constraint

Optical Flow
Estimation

Motion field

When objects move or the camera moves the result is changes in the images.

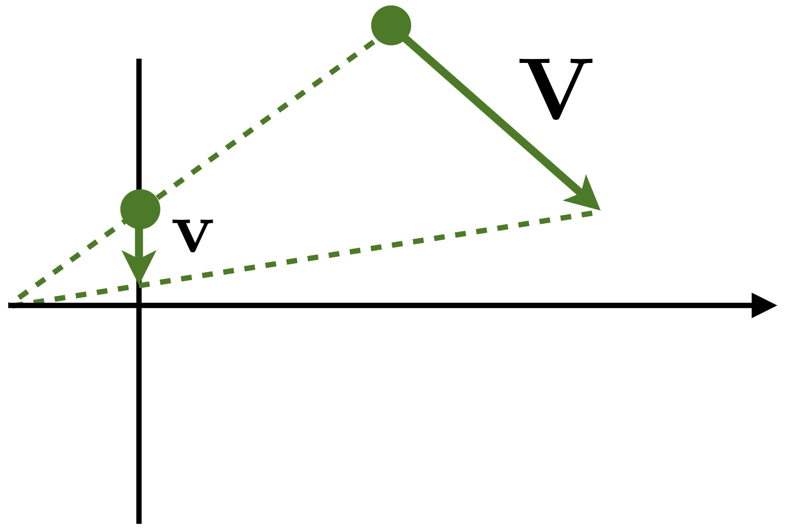
Changes can be used to capture the relative motions as well as the shape of the objects.



$$\mathbf{p} = \Pi(\mathbf{P})$$

Motion field

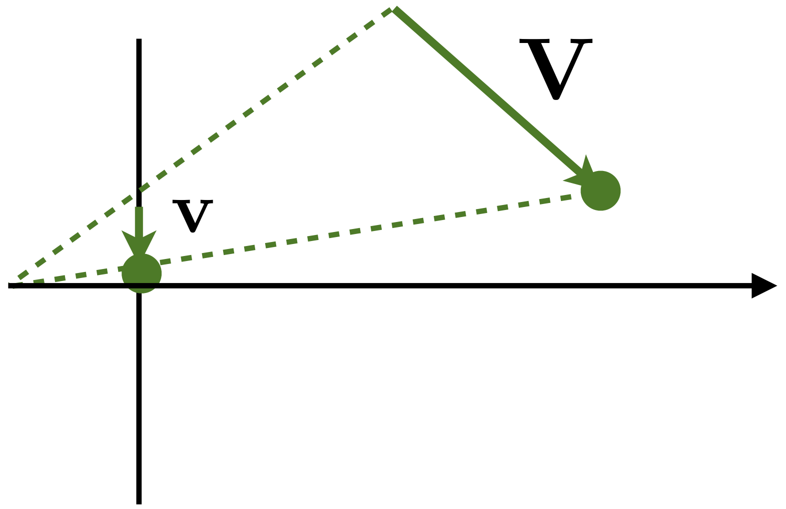
Definition: Motion field assigns a velocity vector to each point in the image according to how the point in 3D moves.



$$\mathbf{v} = \begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \frac{d\mathbf{p}}{dt} = \frac{d\Pi(\mathbf{P})}{dt}$$

Motion field

Definition: Motion field assigns a velocity vector to each point in the image according to how the point in 3D moves.

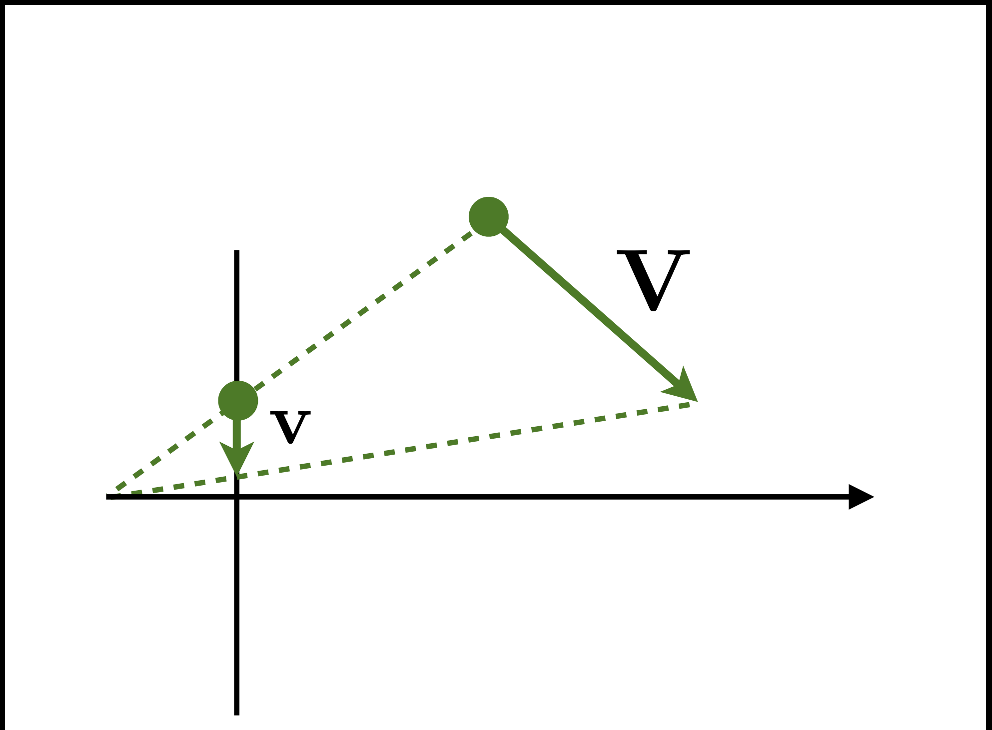


$$\mathbf{v} = \begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \frac{d\mathbf{p}}{dt} = \frac{d\Pi(\mathbf{P})}{dt}$$

Motion field

We **want** the motion field:

- geometric concept

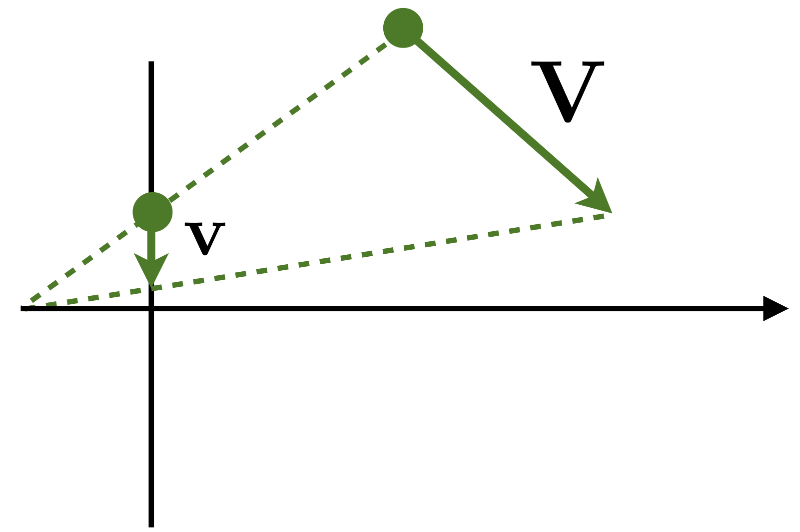


$$\mathbf{v} = \frac{d\mathbf{p}}{dt} = \frac{d\Pi(\mathbf{P})}{dt}$$

Motion field

We **want** the motion field:
- geometric concept

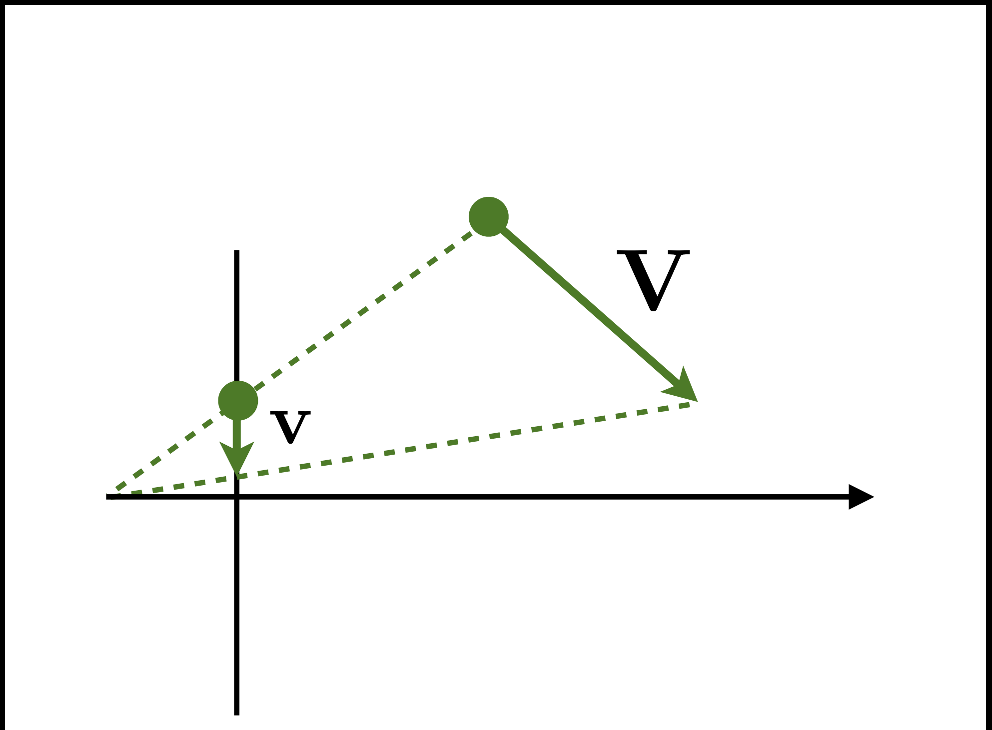
We **have** images:
- photometric concept



$$\mathbf{v} = \frac{d\mathbf{p}}{dt} = \frac{d\Pi(\mathbf{P})}{dt}$$

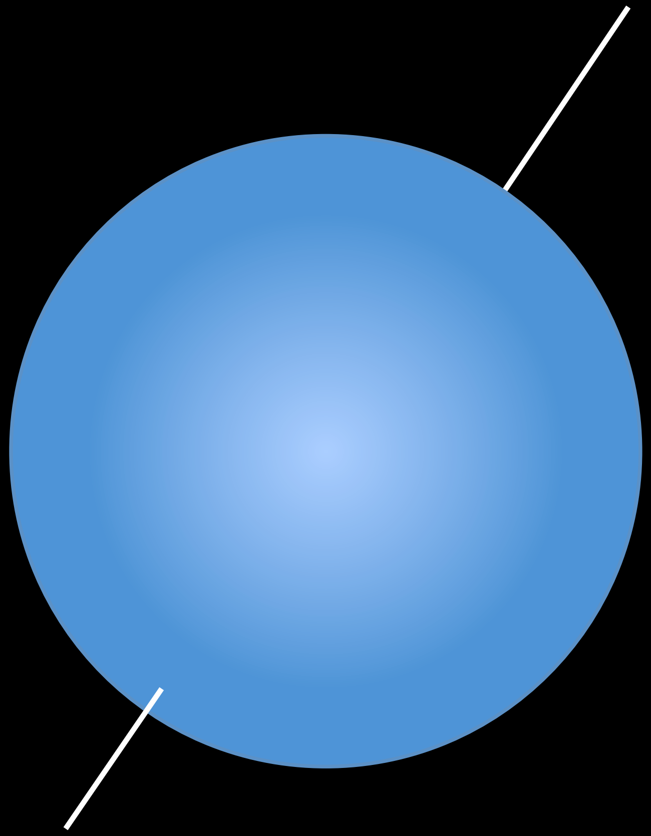
Motion field

Definition: Optical flow is the apparent motion of the brightness pattern.

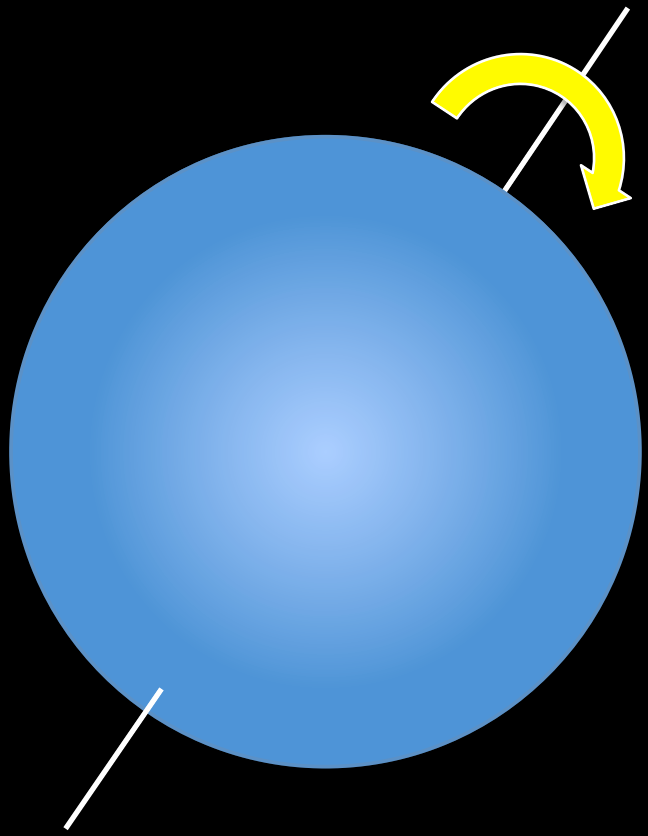


$$\mathbf{v} = \frac{d\mathbf{p}}{dt} = \frac{d\Pi(\mathbf{P})}{dt}$$

Motion Field $\neq$ Optical Flow



Motion Field $\neq$ Optical Flow



Today's agenda

Motion Field
vs.
Optical Flow

Derivation
of
Brightness Constancy
Constraint

Optical Flow
Estimation

Today's agenda

~~Motion Field~~
~~vs.~~
~~Optical Flow~~

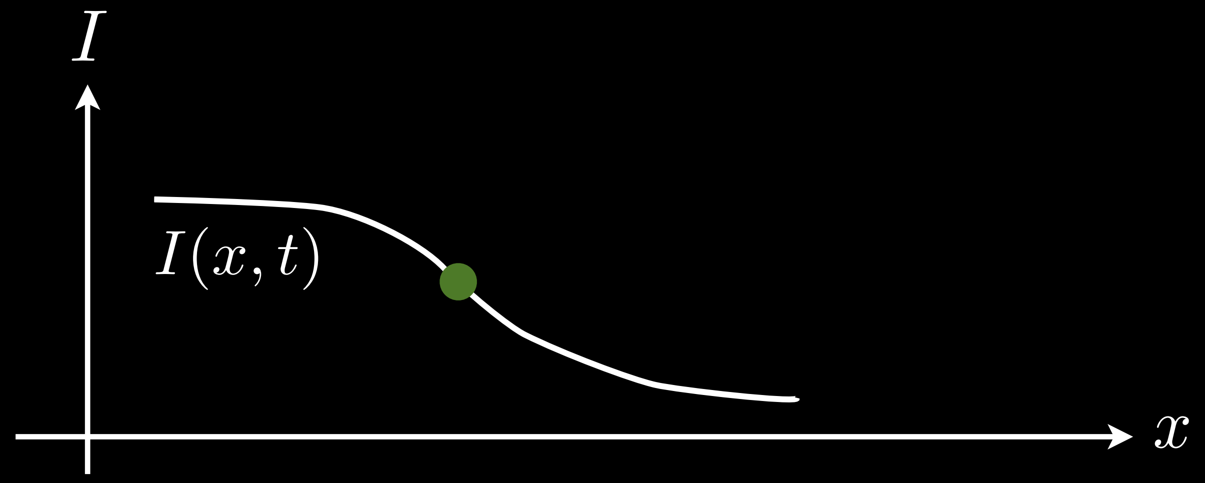
Derivation
of
Brightness Constancy
Constraint

Optical Flow
Estimation

Image brightness of a point remains constant over time

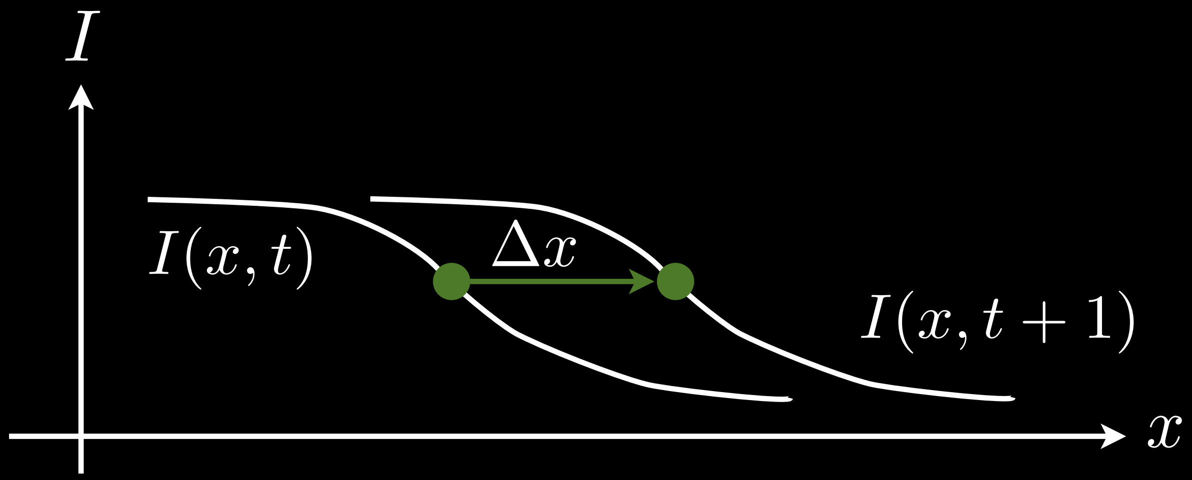
Brightness constancy: Constraint equation

$$I(x + \Delta x, y + \Delta y, t + \Delta t) = I(x, y, t)$$



Brightness constancy: Constraint equation

$$I(x + \Delta x, y + \Delta y, t + \Delta t) = I(x, y, t)$$



Brightness constancy: Constraint equation

$$I(x + \Delta x, y + \Delta y, t + \Delta t) = I(x, y, t)$$

Brightness constancy: Constraint equation

$$I(x+\Delta x, y+\Delta y, t+\Delta t) = I(x,y,t)$$

LHS: Taylor series expansion

Brightness constancy: Constraint equation

$$I(x+\Delta x, y+\Delta y, t+\Delta t) = I(x,y,t)$$

LHS: Taylor series expansion

$$I(x,y,t) + \frac{\partial I}{\partial x}\Delta x + \frac{\partial I}{\partial y}\Delta y + \frac{\partial I}{\partial t}\Delta t + \text{h.o.t.} = I(x,y,t)$$

Brightness constancy: Constraint equation

$$I(x + \Delta x, y + \Delta y, t + \Delta t) = I(x, y, t)$$

LHS: Taylor series expansion

$$\cancel{I(x, y, t)} + \frac{\partial I}{\partial x}\Delta x + \frac{\partial I}{\partial y}\Delta y + \frac{\partial I}{\partial t}\Delta t + \text{h.o.t.} = \cancel{I(x, y, t)}$$

cancel

Brightness constancy: Constraint equation

$$I(x+\Delta x, y+\Delta y, t+\Delta t) = I(x,y,t)$$

LHS: Taylor series expansion

$$\cancel{I(x,y,t)} + \frac{\partial I}{\partial x}\Delta x + \frac{\partial I}{\partial y}\Delta y + \frac{\partial I}{\partial t}\Delta t + \text{h.o.t.} = \cancel{I(x,y,t)}$$

cancel

$$\frac{\partial I}{\partial x}\Delta x + \frac{\partial I}{\partial y}\Delta y + \frac{\partial I}{\partial t}\Delta t + \text{h.o.t.} = 0$$

Brightness constancy: Constraint equation

$$\frac{\partial I}{\partial x}\Delta x + \frac{\partial I}{\partial y}\Delta y + \frac{\partial I}{\partial t}\Delta t + \text{h.o.t.} = 0$$

Brightness constancy: Constraint equation

$$\frac{\partial I}{\partial x}\Delta x + \frac{\partial I}{\partial y}\Delta y + \frac{\partial I}{\partial t}\Delta t + \mathrm{h.o.t.} = 0$$

divide by Δt

Brightness constancy: Constraint equation

$$\frac{\partial I}{\partial x}\Delta x + \frac{\partial I}{\partial y}\Delta y + \frac{\partial I}{\partial t}\Delta t + \text{h.o.t.} = 0$$

divide by Δt

$$\frac{\partial I}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial I}{\partial y}\frac{\Delta y}{\Delta t} + \frac{\partial I}{\partial t}\frac{\Delta t}{\Delta t} + \text{h.o.t.} = 0$$

Brightness constancy: Constraint equation

$$\frac{\partial I}{\partial x}\Delta x + \frac{\partial I}{\partial y}\Delta y + \frac{\partial I}{\partial t}\Delta t + \mathrm{h.o.t.} = 0$$

divide by Δt

$$\frac{\partial I}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial I}{\partial y}\frac{\Delta y}{\Delta t} + \frac{\partial I}{\partial t}\xcancel{\frac{\Delta t}{\Delta t}} + \mathrm{h.o.t.} = 0$$

Brightness constancy: Constraint equation

$$\frac{\partial I}{\partial x}\Delta x + \frac{\partial I}{\partial y}\Delta y + \frac{\partial I}{\partial t}\Delta t + \text{h.o.t.} = 0$$

divide by Δt

$$\frac{\partial I}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial I}{\partial y}\frac{\Delta y}{\Delta t} + \frac{\partial I}{\partial t}\cancel{\frac{\Delta t}{\Delta t}} + \text{h.o.t.} = 0$$

$$\Delta t \to 0$$

Brightness constancy: Constraint equation

$$\frac{\partial I}{\partial x}\Delta x + \frac{\partial I}{\partial y}\Delta y + \frac{\partial I}{\partial t}\Delta t + \text{h.o.t.} = 0$$

divide by Δt

$$\frac{\partial I}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial I}{\partial y}\frac{\Delta y}{\Delta t} + \frac{\partial I}{\partial t}\xcancel{\frac{\Delta t}{\Delta t}} + \text{h.o.t.} = 0$$

$\Delta t \rightarrow 0$

$$\frac{\partial I}{\partial x}\frac{dx}{dt} + \frac{\partial I}{\partial y}\frac{dy}{dt} + \frac{\partial I}{\partial t} = 0$$

Wednesday, February 22, 2012

Brightness constancy: Constraint equation

$$\frac{\partial I}{\partial x}\frac{dx}{dt} + \frac{\partial I}{\partial y}\frac{dy}{dt} + \frac{\partial I}{\partial t} = 0$$

Brightness constancy: Constraint equation

$$\frac{\partial I}{\partial x}\boxed{\frac{dx}{dt}}+\frac{\partial I}{\partial y}\boxed{\frac{dy}{dt}}+\frac{\partial I}{\partial t}=0$$

Brightness constancy: Constraint equation

$$\frac{\partial I}{\partial x}\boxed{\frac{dx}{dt}}+\frac{\partial I}{\partial y}\boxed{\frac{dy}{dt}}+\frac{\partial I}{\partial t}=0$$

$$\frac{dx}{dt}=u,\ \frac{dy}{dt}=v$$

Brightness constancy: Constraint equation

$$\frac{\partial I}{\partial x}\boxed{\frac{dx}{dt}} + \frac{\partial I}{\partial y}\boxed{\frac{dy}{dt}} + \frac{\partial I}{\partial t} = 0$$

$$\frac{dx}{dt} = u, \;\; \frac{dy}{dt} = v$$

$$\frac{\partial I}{\partial x}u + \frac{\partial I}{\partial y}v + \frac{\partial I}{\partial t} = 0$$

Brightness constancy: Constraint equation

$$\frac{\partial I}{\partial x}\frac{dx}{dt}+\frac{\partial I}{\partial y}\frac{dy}{dt}+\frac{\partial I}{\partial t}=0$$

$$\frac{dx}{dt}=u,\;\;\frac{dy}{dt}=v$$

$$\frac{\partial I}{\partial x}u+\frac{\partial I}{\partial y}v+\frac{\partial I}{\partial t}=0$$

Brightness Constancy Constraint

$$I_x u+I_y v+I_t=0$$

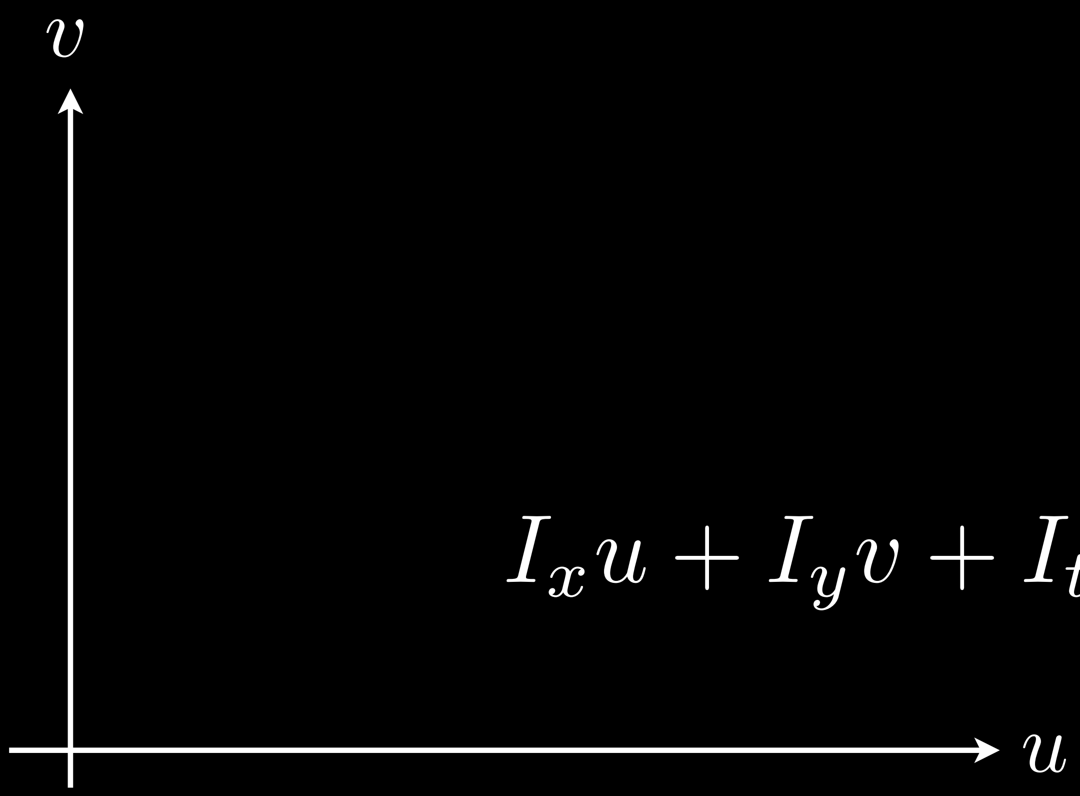
Brightness
Constancy Constraint

$$I_x u + I_y v + I_t = 0$$

How do we recover the velocity?

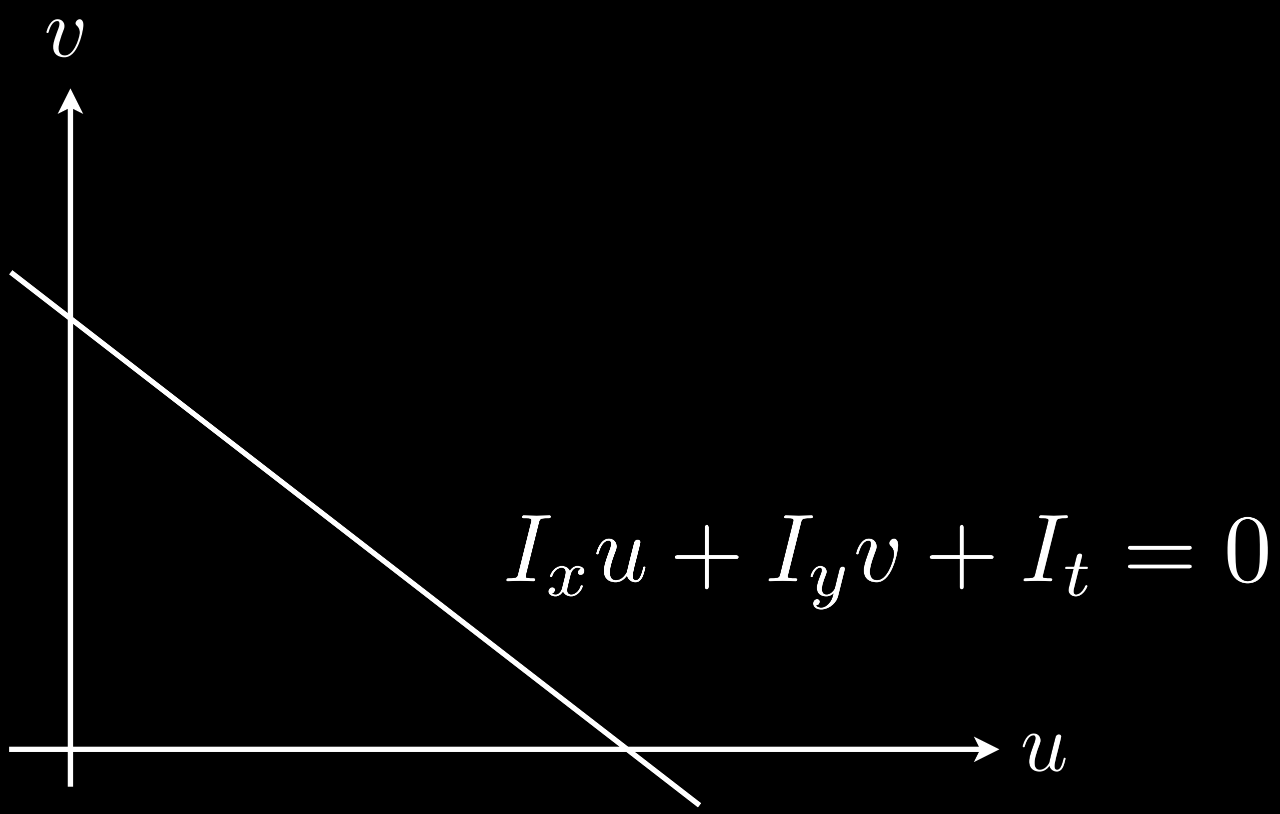
Brightness Constancy Constraint

$$I_x u + I_y v + I_t = 0$$



Brightness Constancy Constraint

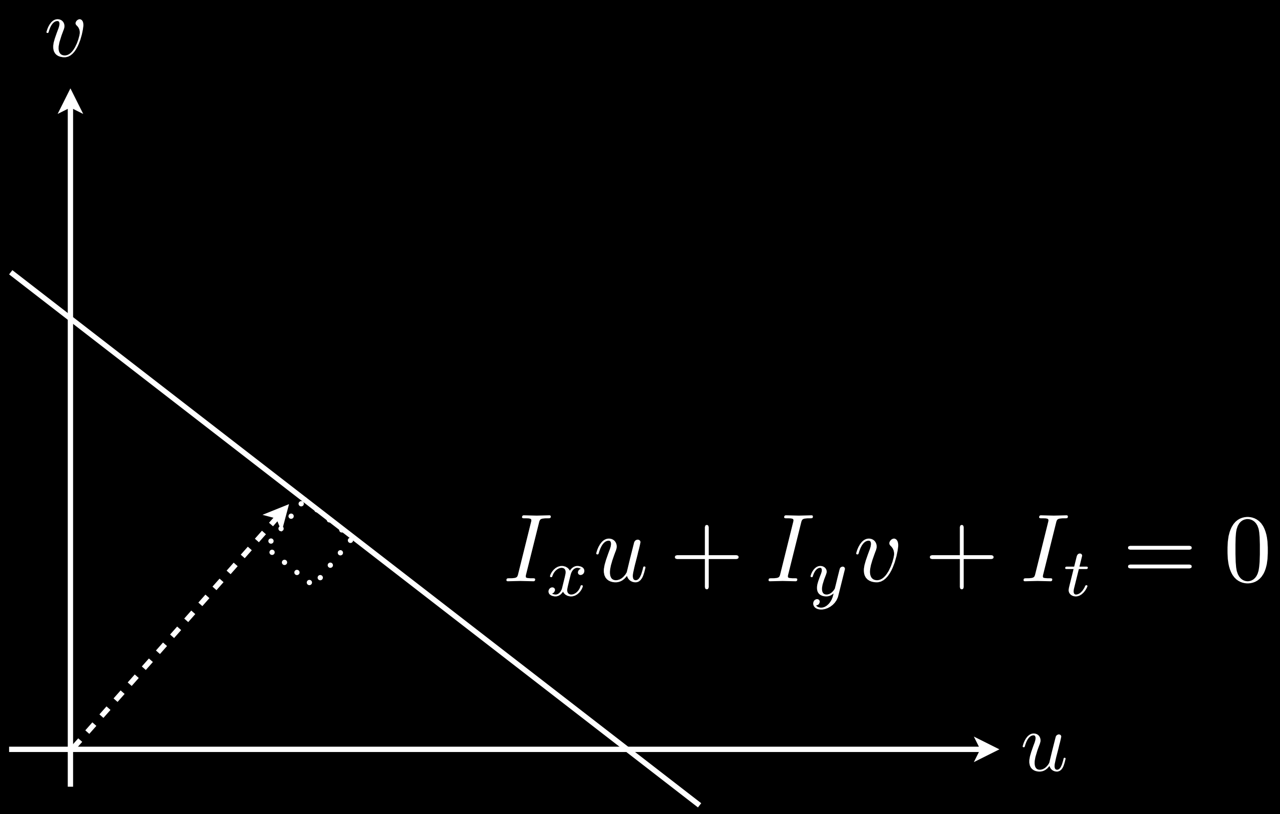
$$I_x u + I_y v + I_t = 0$$



Brightness Constancy Constraint

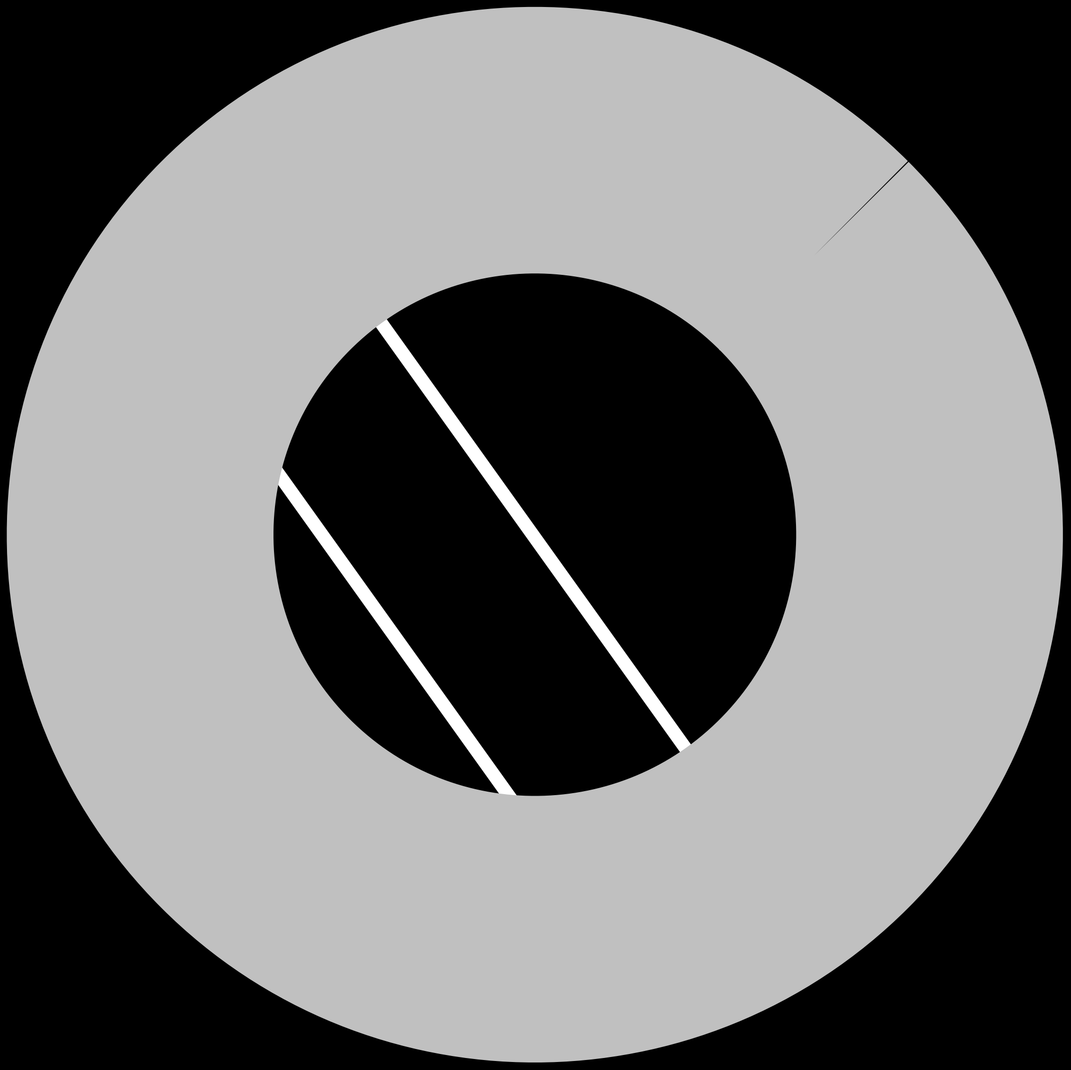
$$I_x u + I_y v + I_t = 0$$

$$\mathbf{v}_n = -\frac{I_t}{\|\nabla I\|} \frac{\nabla I}{\|\nabla I\|}$$

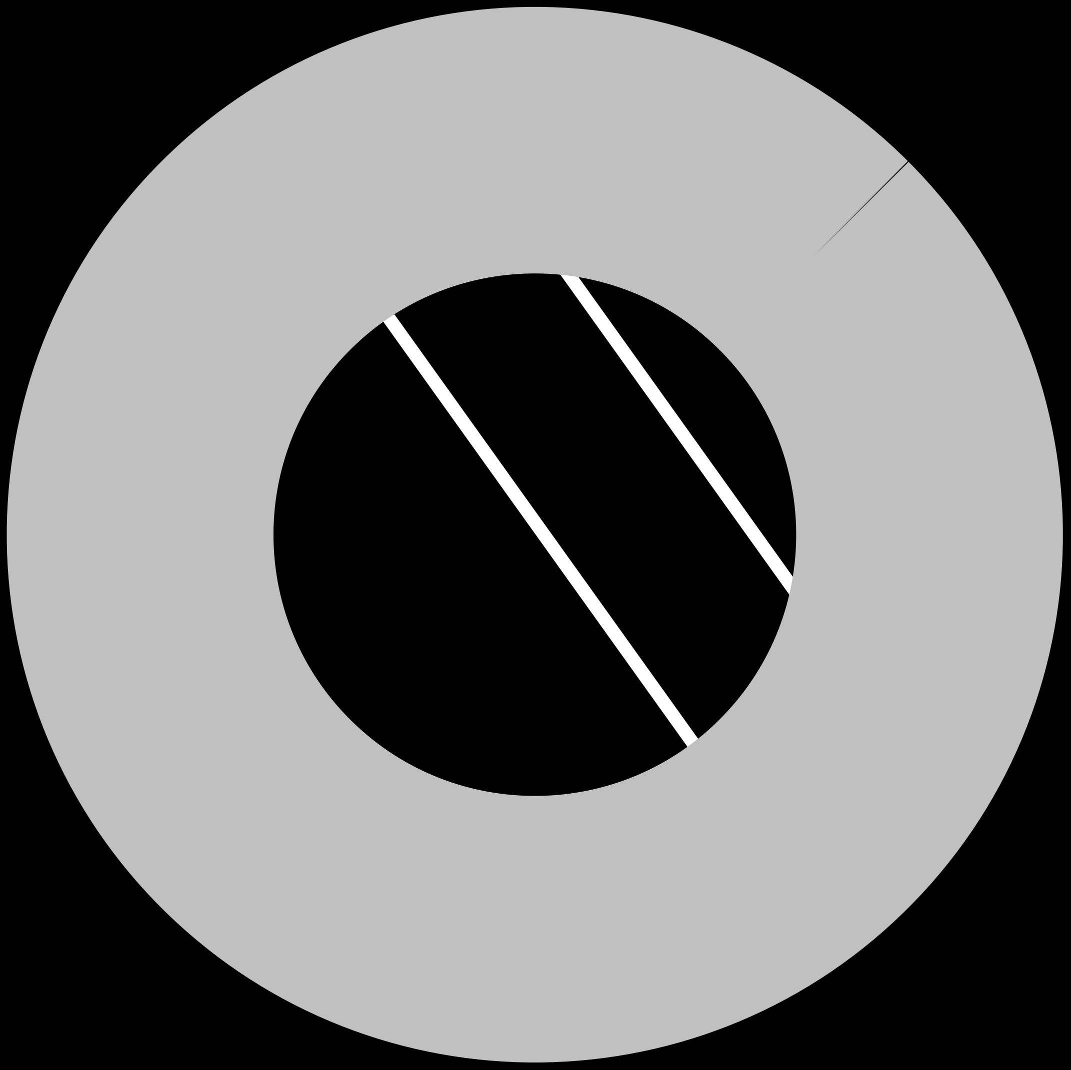


Aperture Problem

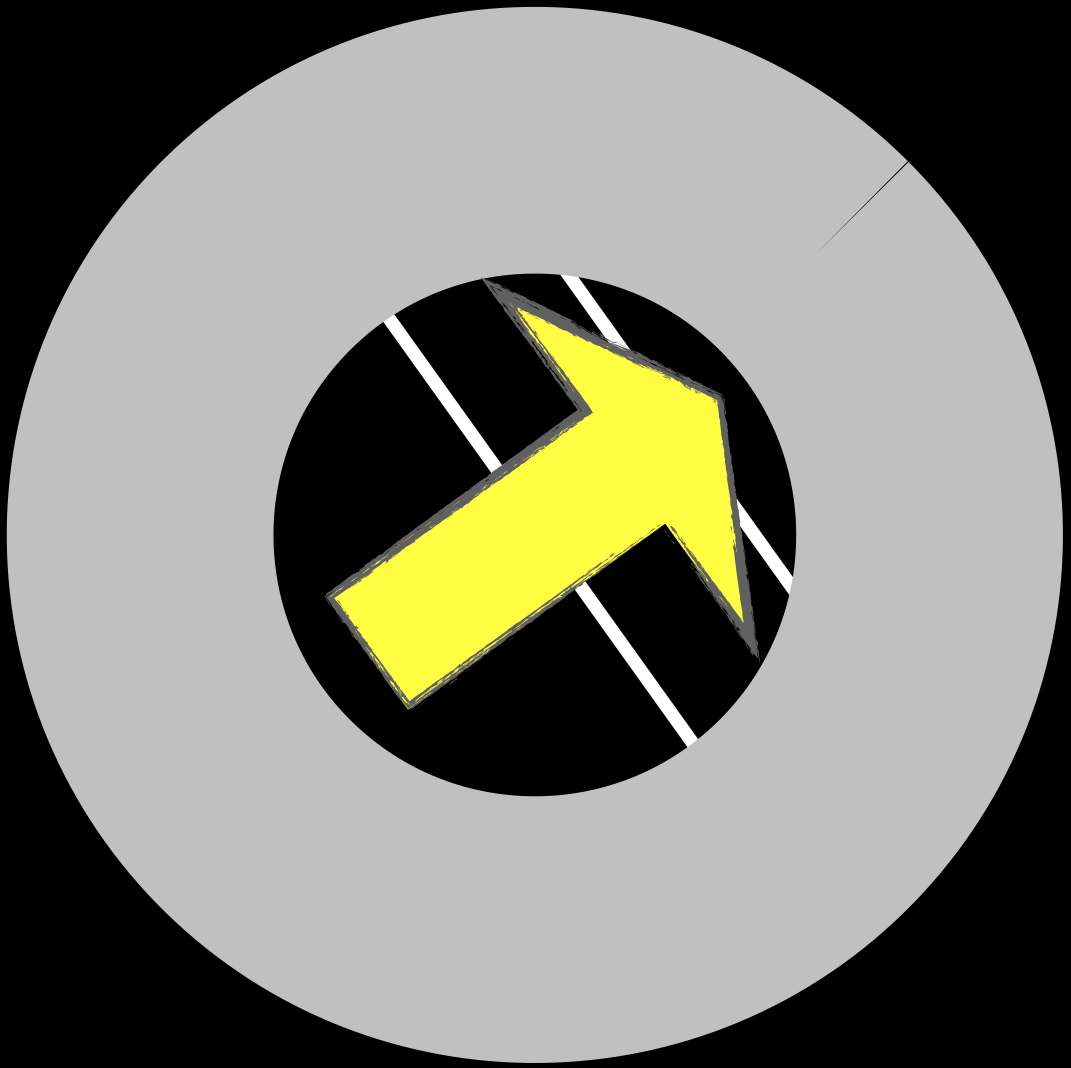
Aperture Problem



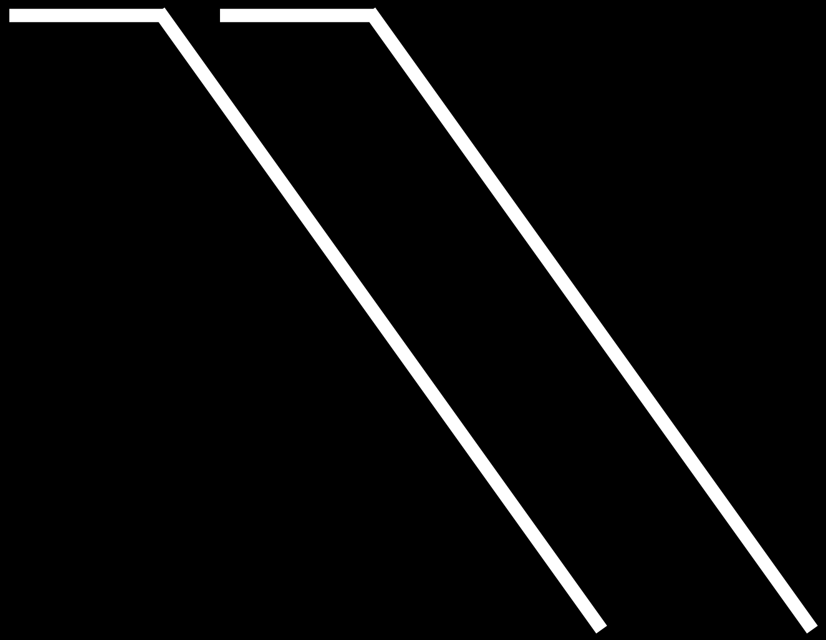
Aperture Problem



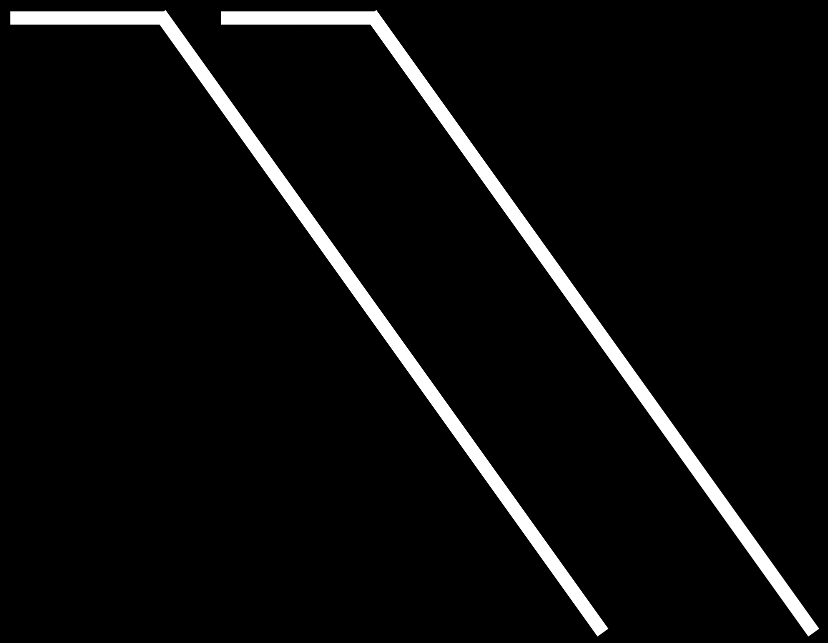
Aperture Problem



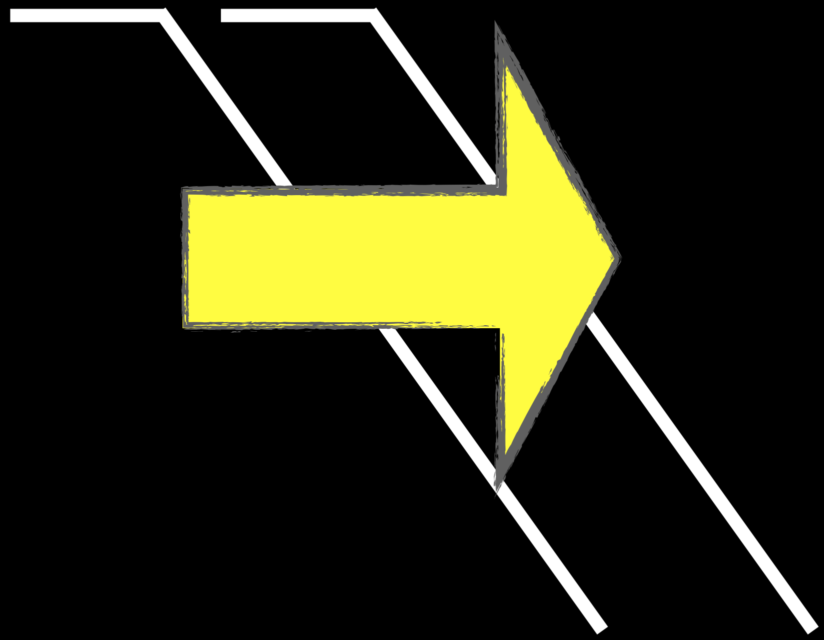
Aperture Problem



Aperture Problem



Aperture Problem



Solution: Impose additional constraints

Today's agenda

~~Motion Field~~
~~vs.~~
~~Optical Flow~~

Derivation
of
Brightness Constancy
Constraint

Optical Flow
Estimation

Today’s agenda

~~Motion Field~~
~~vs.~~
~~Optical Flow~~

~~Derivation~~
~~of~~
~~Brightness Constancy~~
~~Constraint~~

Optical Flow
Estimation


Local
Global

Local

Local: Lucas-Kanade

IJCA 1981

An Iterative Image Registration Technique with an Application to Stereo Vision

Bruce D. Lucas
Takeo Kanade

Computer Science Department
Carnegie-Mellon University
Pittsburgh, Pennsylvania 15213

Abstract

Image registration finds a variety of applications in computer vision. Unfortunately, traditional image registration techniques tend to be costly. We *present a new image registration technique that makes use of the spatial intensity gradient of the images to find a good match using a type of Newton-Raphson iteration. Our technique is taster because it examines far fewer potential matches between the images than existing techniques Furthermore, this registration technique can be generalized to handle rotation, scaling and shearing. We show how our technique can be adapted tor use in a stereo vision system.*

1. Introduction

Image registration finds a variety of applications in computer vision, such as image matching for stereo vision, pattern recognition, and motion analysis. Unfortunately, existing techniques for image registration tend to be costly. Moreover, they generally fail to deal with rotation or other distortions of the images.

In this paper we present a new image registration technique that uses spatial intensity gradient information to direct the search for the position that yields the best match. By taking more information about the images into account, this technique is able to find the best match between two images with far fewer comparisons of images than techniques that examine the possible positions of registration in some fixed order. Our technique takes advantage of the fact that in many applications the two images are already in approximate registration. This technique can be generalized to deal with arbitrary linear distortions of the image, including rotation. We then describe a stereo vision system that uses this registration technique, and suggest some further avenues tor research toward making effective use of this method in stereo image understanding.

2. The registration problem

The translational image registration problem can be characterized as follows: We are given functions *F(x)* and G(x) which give the respective pixel values at each location x in two images, where x is a vector. We wish to find the disparity vector h that minimizes some measure of the difference between F(x + h) and G(x), for x in some region of interest *R*. (See figure 1)

Figure 1: The image registration problem

Typical measures of the difference between $F(x + h)$ and $G(x)$ are:

- L_1 norm $= \sum_{x \in R} |F(x + h) - G(x)|$
- L_2 norm $= \left(\sum_{x \in R} [F(x + h) - G(x)]^2\right)^{1/2}$
- negative of normalized correlation

$$= \frac{-\sum_{x \in R} F(x + h)G(x)}{\left(\sum_{x \in R} F(x + h)^2\right)^{1/2}\left(\sum_{x \in R} G(x)^2\right)^{1/2}}$$

We will propose a more general measure of image difference, of which both the L_2 norm and the correlation are special cases. The L_1 norm is chiefly of interest as an inexpensive approximation to the L_2 norm.

3. Existing techniques

An obvious technique for registering two images is to calculate a measure of the difference between the images at all possible values of the disparity vector h—that is, to exhaustively search the space of possible values of h. This technique is *very* time consuming; if the size of the picture G(x) is *NxN,* and the region of

---

This research was sponsored by the Defense Advanced Research Projects Agency (DOD), ARPA Order No. 3597, monitored by the Air Force Avionics Laboratory Under Contract F33615 78-C-1551.

Local: Lucas-Kanade

5632 citations !!!

IJCA 1981

An Iterative Image Registration Technique with an Application to Stereo Vision

Bruce D. Lucas
Takeo Kanade

Computer Science Department
Carnegie-Mellon University
Pittsburgh, Pennsylvania 15213

Abstract

Image registration finds a variety of applications in computer vision. Unfortunately, traditional image registration techniques tend to be costly. We present a new image registration technique that makes use of the spatial intensity gradient of the images to find a good match using a type of Newton-Raphson iteration. Our technique is faster because it examines far fewer potential matches between the images than existing techniques Furthermore, this registration technique can be generalized to handle rotation, scaling and shearing. We show how our technique can be adapted for use in a stereo vision system.

1. Introduction

Image registration finds a variety of applications in computer vision, such as image matching for stereo vision, pattern recognition, and motion analysis. Unfortunately, existing techniques for image registration tend to be costly. Moreover, they generally fail to deal with rotation or other distortions of the images.

In this paper we present a new image registration technique that uses spatial intensity gradient information to direct the search for the position that yields the best match. By taking more information about the images into account, this technique is able to find the best match between two images with far fewer comparisons of images than techniques that examine the possible positions of registration in some fixed order. Our technique takes advantage of the fact that in many applications the two images are already in approximate registration. This technique can be generalized to deal with arbitrary linear distortions of the image, including rotation. We then describe a stereo vision system that uses this registration technique, and suggest some further avenues for research toward making effective use of this method in stereo image understanding.

2. The registration problem

The translational image registration problem can be characterized as follows: We are given functions *F(x)* and G(x) which give the respective pixel values at each location x in two images, where x is a vector. We wish to find the disparity vector h that minimizes some measure of the difference between F(x + h) and G(x), for x in some region of interest *R*. (See figure 1)

Figure 1: The image registration problem

Typical measures of the difference between $F(\mathbf{x}+\mathbf{h})$ and $G(\mathbf{x})$ are:

- L_1 norm $= \sum_{x \in R} |F(x+h) - G(x)|$
- L_2 norm $= \left(\sum_{x \in R} [F(x+h) - G(x)]^2\right)^{1/2}$
- negative of normalized correlation

$$= \frac{-\sum_{x \in R} F(x+h)G(x)}{\left(\sum_{x \in R} F(x+h)^2\right)^{1/2}\left(\sum_{x \in R} G(x)^2\right)^{1/2}}$$

We will propose a more general measure of image difference, of which both the L_2 norm and the correlation are special cases. The L_1 norm is chiefly of interest as an inexpensive approximation to the L_2 norm.

3. Existing techniques

An obvious technique for registering two images is to calculate a measure of the difference between the images at all possible values of the disparity vector h—that is, to exhaustively search the space of possible values of h. This technique is *very* time consuming; if the size of the picture G(x) is *NxN*, and the region of

---

This research was sponsored by the Defense Advanced Research Projects Agency (DOD), ARPA Order No 3597, monitored by the Air Force Avionics Laboratory Under Contract F33615 78-C-1551.

Assume velocity is constant within pixel's neighbourhood

Lucas-Kanade flow

$$\begin{pmatrix} I_x(\mathbf{p}_1) & I_y(\mathbf{p}_1) \\ I_x(\mathbf{p}_2) & I_y(\mathbf{p}_2) \\ \vdots & \vdots \\ I_x(\mathbf{p}_N) & I_y(\mathbf{p}_N) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = - \begin{pmatrix} I_t(\mathbf{p}_1) \\ I_t(\mathbf{p}_2) \\ \vdots \\ I_t(\mathbf{p}_N) \end{pmatrix}$$

$$\mathbf{A}_{N^2 \times 2} \qquad \mathbf{v}_{2 \times 1} \qquad \mathbf{b}_{N^2 \times 1}$$

overdetermined system

Least-squares

$$\underset{\mathbf{v}}{\operatorname{argmin}} \|\mathbf{A}\mathbf{v} - \mathbf{b}\|^2$$

Least-squares

$$\underset{\mathbf{v}}{\operatorname{argmin}} \|\mathbf{A}\mathbf{v} - \mathbf{b}\|^2$$

$$\mathbf{A}^\top \mathbf{A}\mathbf{v} = \mathbf{A}^\top \mathbf{b}$$

Least-squares

$$\underset{\mathbf{v}}{\operatorname{argmin}} \|\mathbf{A}\mathbf{v} - \mathbf{b}\|^2$$

$$\mathbf{A}^\top \mathbf{A}\mathbf{v} = \mathbf{A}^\top \mathbf{b}$$

$$\underbrace{\begin{pmatrix} \sum I_x^2 & \sum I_x I_y \\ \sum I_x I_y & \sum I_y^2 \end{pmatrix}}_{\mathbf{A}^\top \mathbf{A}_{2\times 2}} \underbrace{\begin{pmatrix} u \\ v \end{pmatrix}}_{\mathbf{v}_{2\times 1}} = - \underbrace{\begin{pmatrix} \sum I_x I_t \\ \sum I_y I_t \end{pmatrix}}_{\mathbf{A}^\top \mathbf{b}_{2\times 1}}$$

Least-squares

$$\underset{\mathbf{v}}{\operatorname{argmin}} \|\mathbf{A}\mathbf{v} - \mathbf{b}\|^2$$

$$\mathbf{A}^\top \mathbf{A} \mathbf{v} = \mathbf{A}^\top \mathbf{b}$$

$$\underbrace{\begin{pmatrix} \sum I_x^2 & \sum I_x I_y \\ \sum I_x I_y & \sum I_y^2 \end{pmatrix}}_{\mathbf{A}^\top \mathbf{A}_{2\times 2}} \underbrace{\begin{pmatrix} u \\ v \end{pmatrix}}_{\mathbf{v}_{2\times 1}} = - \underbrace{\begin{pmatrix} \sum I_x I_t \\ \sum I_y I_t \end{pmatrix}}_{\mathbf{A}^\top \mathbf{b}_{2\times 1}}$$

$$\mathbf{v} = (\mathbf{A}^\top \mathbf{A})^{-1} \mathbf{A}^\top \mathbf{b}$$

Least-squares

$$\underset{\mathbf{v}}{\operatorname{argmin}} \|\mathbf{A}\mathbf{v} - \mathbf{b}\|^2$$

$$\mathbf{A}^\top \mathbf{A} \mathbf{v} = \mathbf{A}^\top \mathbf{b}$$

$$\underset{\mathbf{A}^\top\mathbf{A}_{2\times 2}}{\begin{pmatrix} \sum I_x^2 & \sum I_x I_y \\ \sum I_x I_y & \sum I_y^2 \end{pmatrix}} \underset{\mathbf{v}_{2\times 1}}{\begin{pmatrix} u \\ v \end{pmatrix}} = - \underset{\mathbf{A}^\top\mathbf{b}_{2\times 1}}{\begin{pmatrix} \sum I_x I_t \\ \sum I_y I_t \end{pmatrix}}$$

$$\mathbf{v} = (\mathbf{A}^\top \mathbf{A})^{-1} \mathbf{A}^\top \mathbf{b}$$

Harris detector: Derivation

$$E(\Delta x, \Delta y) = \begin{pmatrix} \Delta x & \Delta y \end{pmatrix} \mathbf{M} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$$

where

$$\mathbf{M} = \sum_{x,y} w(x,y) \begin{pmatrix} I_x^2 & I_x I_y \\ I_x I_y & I_y^2 \end{pmatrix}$$

$\mathbf{M}$ captures the variation of the gradients within the local patch

Least-squares

$$\underset{\mathbf{v}}{\operatorname{argmin}} \|\mathbf{A}\mathbf{v} - \mathbf{b}\|^2$$

$$\mathbf{A}^\top \mathbf{A}\mathbf{v} = \mathbf{A}^\top \mathbf{b}$$

$$\underset{\mathbf{A}^\top \mathbf{A}_{2\times 2}}{\begin{pmatrix} \sum I_x^2 & \sum I_x I_y \\ \sum I_x I_y & \sum I_y^2 \end{pmatrix}} \underset{\mathbf{v}_{2\times 1}}{\begin{pmatrix} u \\ v \end{pmatrix}} = - \underset{\mathbf{A}^\top \mathbf{b}_{2\times 1}}{\begin{pmatrix} \sum I_x I_t \\ \sum I_y I_t \end{pmatrix}}$$

$$\mathbf{v} = (\mathbf{A}^\top \mathbf{A})^{-1} \mathbf{A}^\top \mathbf{b}$$

$$\mathbf{v} = (\mathbf{A}^\top \mathbf{A})^{-1} \mathbf{A}^\top \mathbf{b}$$

Conditions for solvability?

$$\mathbf{v} = (\mathbf{A}^\top \mathbf{A})^{-1} \mathbf{A}^\top \mathbf{b}$$

Conditions for solvability?

- $\mathbf{A}^\top \mathbf{A}$ is invertible

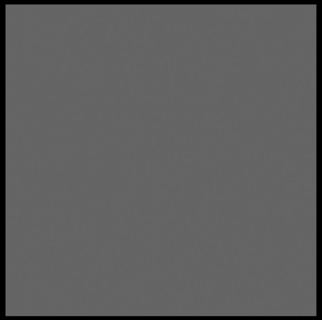
$$\mathbf{v} = (\mathbf{A}^\top \mathbf{A})^{-1} \mathbf{A}^\top \mathbf{b}$$

Conditions for solvability?

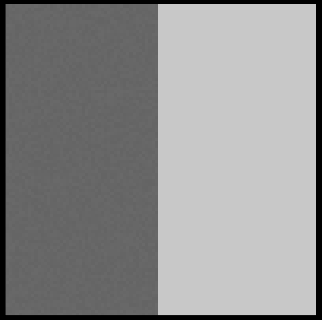
- $\mathbf{A}^\top \mathbf{A}$ is invertible
- $\mathbf{A}^\top \mathbf{A}$ eigenvalues $\lambda_1, \lambda_2 \gg 0$
- $\mathbf{A}^\top \mathbf{A}$ should be well conditioned
  - λ_1 / λ_2 not too large

Conditions for solvability

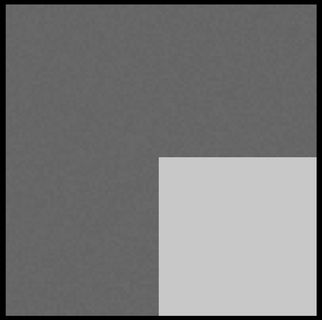
- $\mathbf{A}^\top \mathbf{A}$ is invertible
- $\mathbf{A}^\top \mathbf{A}$ eigenvalues $\lambda_1, \lambda_2 \gg 0$
- $\mathbf{A}^\top \mathbf{A}$ should be well conditioned
  - λ_1 / λ_2 not too large



“Flat”



“Edge”



“Corner”

Parametric flow

$$I(x + \Delta x, y + \Delta y, t + \Delta t) = I(x, y, t)$$

Parametric flow

$$I(x + \Delta x(\mathbf{x};\mathbf{p}), y + \Delta y(\mathbf{x};\mathbf{p}), t + \Delta t) = I(x, y, t)$$

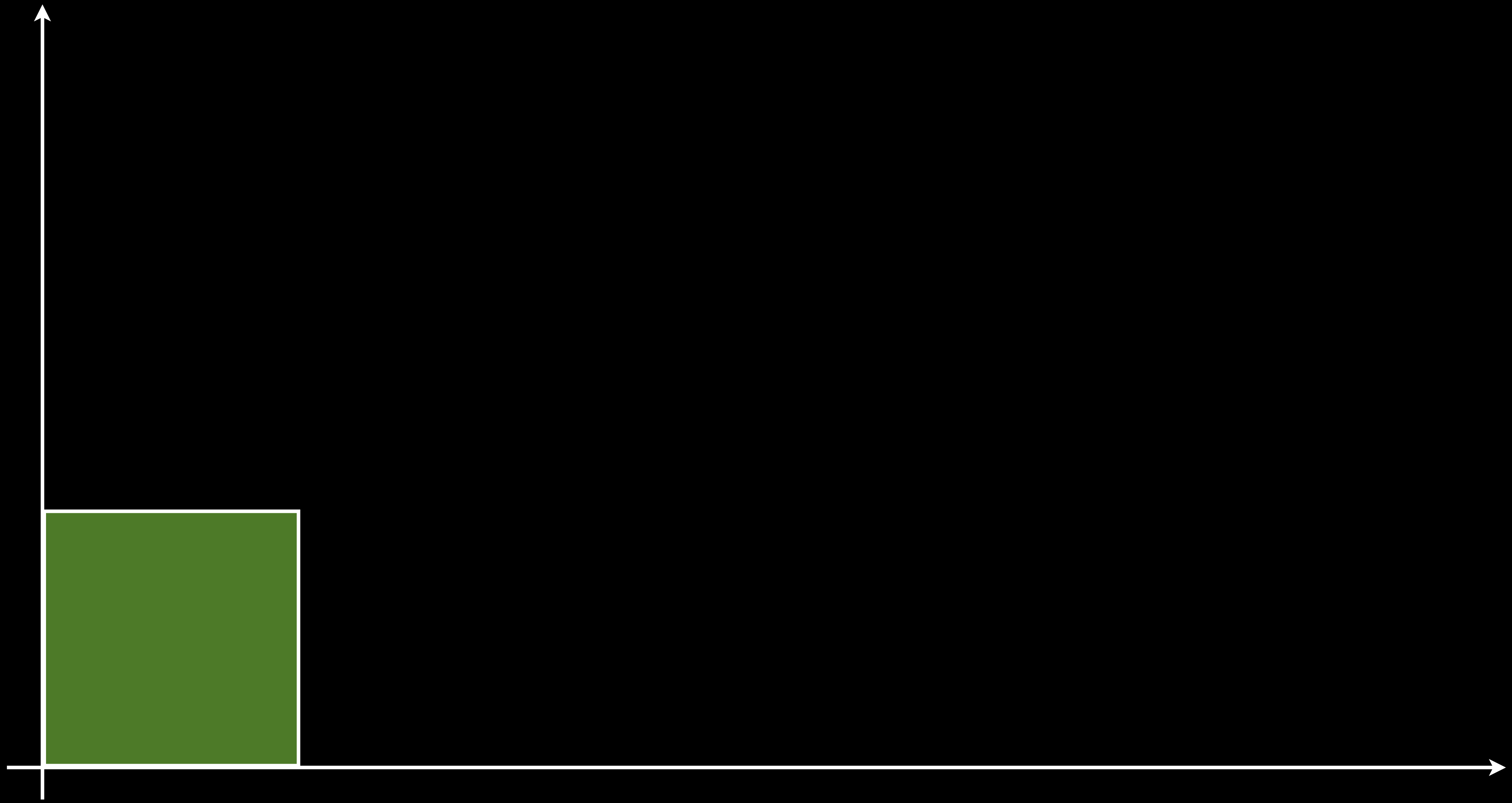
Parametric flow

$$I(x + \Delta x(\mathbf{x};\mathbf{p}), y + \Delta y(\mathbf{x};\mathbf{p}), t + \Delta t) = I(x, y, t)$$

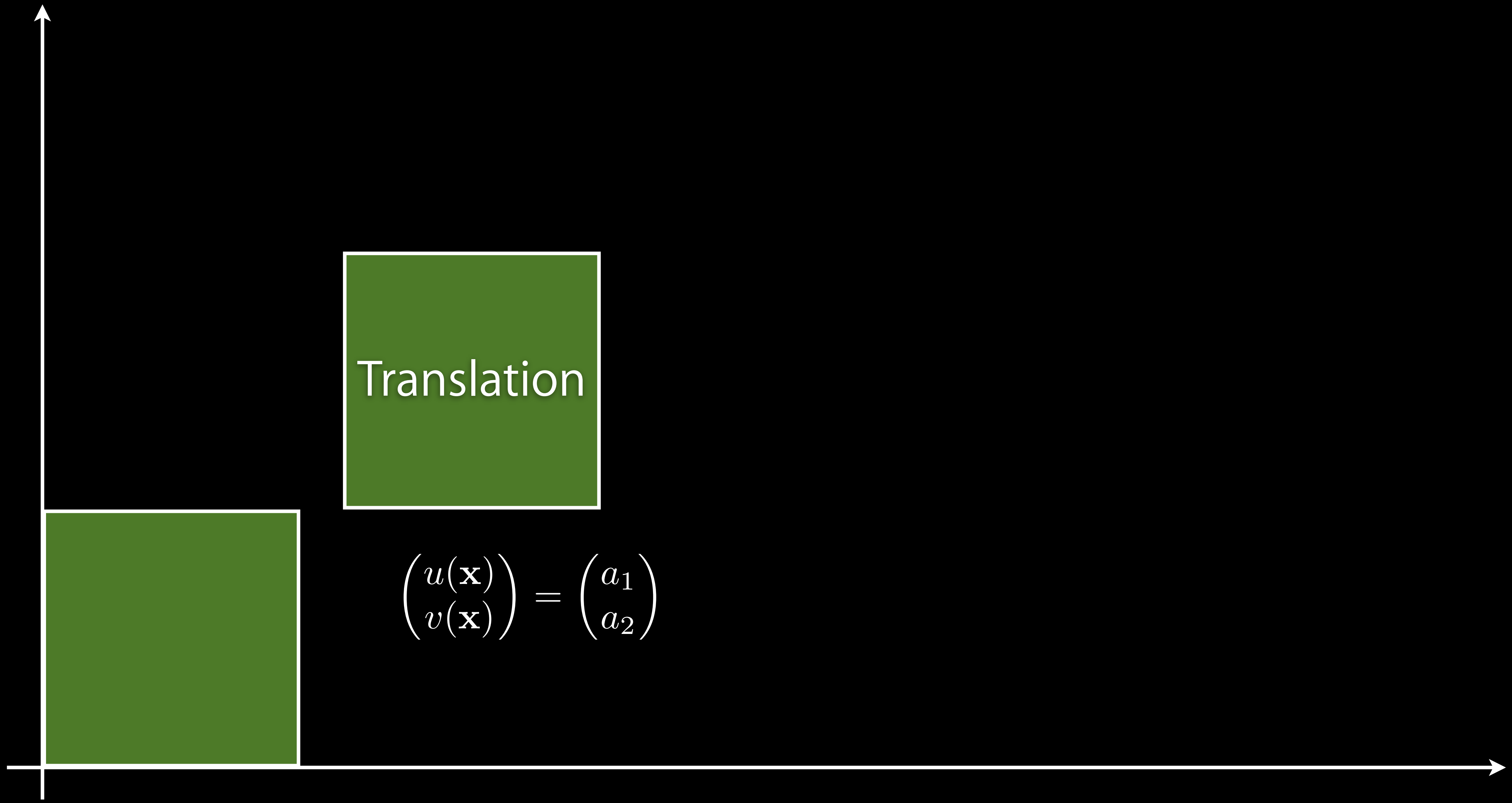
similar derivation as constant velocity

www.cse.yorku.ca/~kosta/CompVis_Notes/optical_affine_flow_computation.pdf

Parametric motion examples:

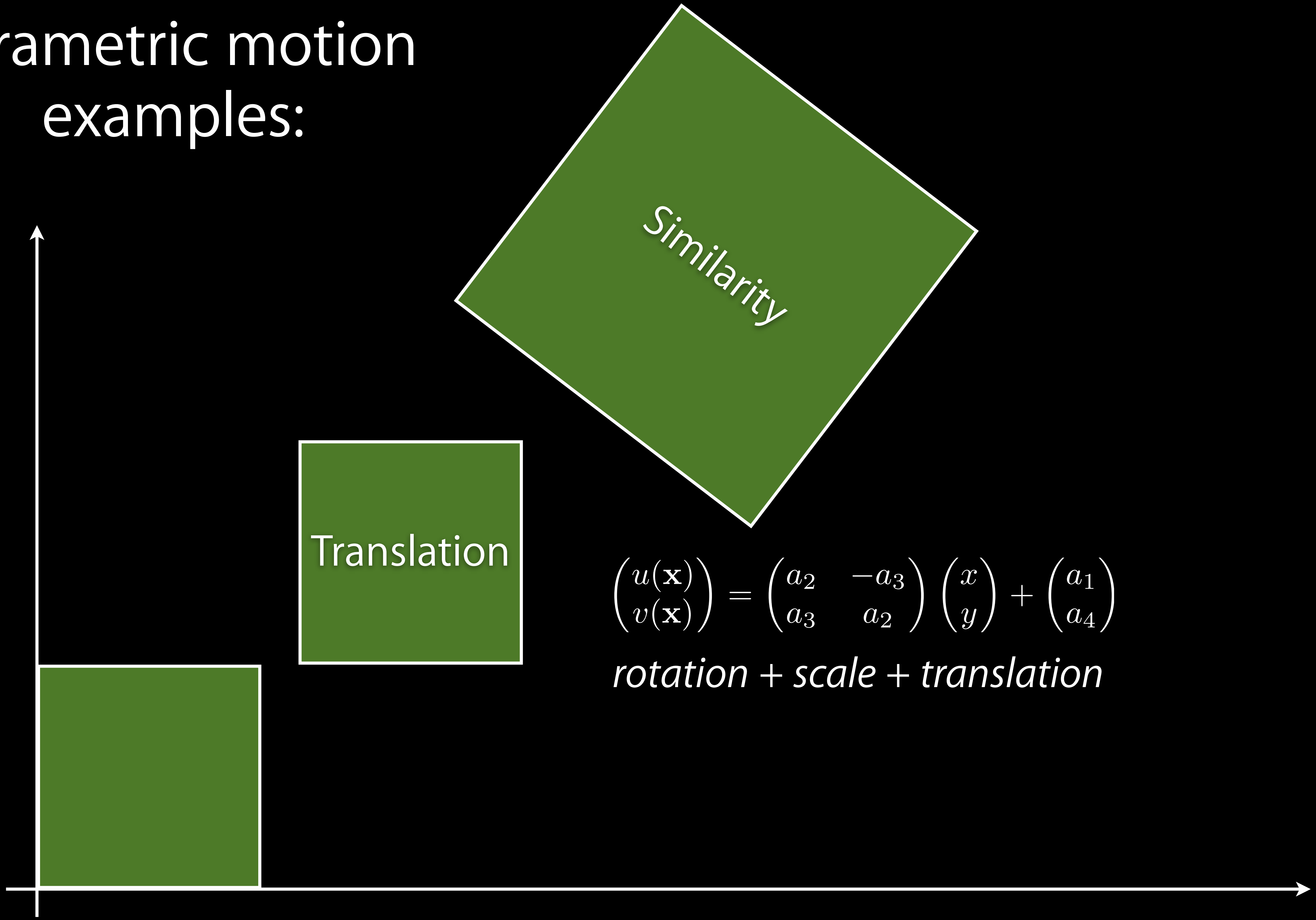


Parametric motion examples:



$$\begin{pmatrix} u(\mathbf{x}) \\ v(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

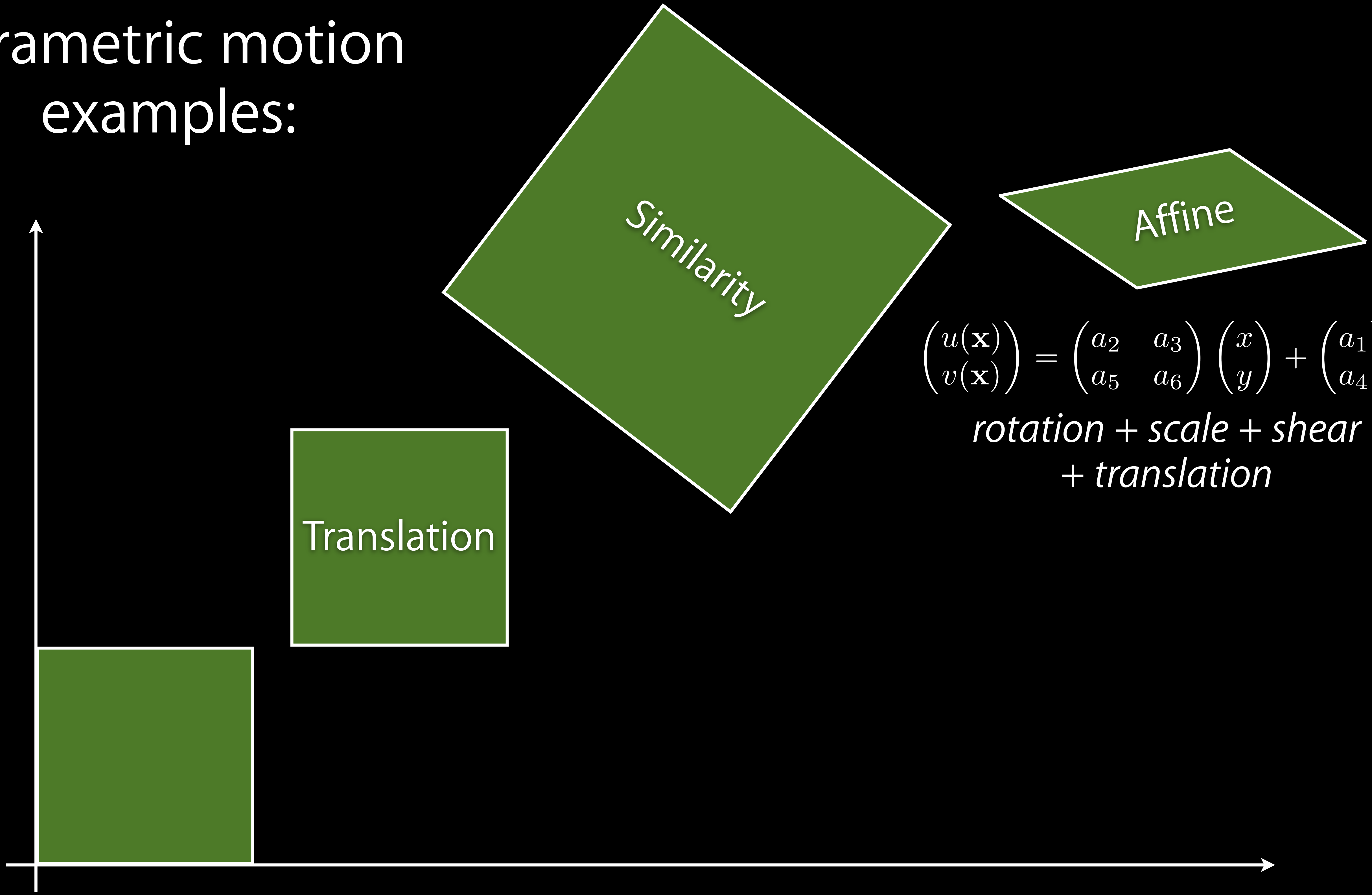
Parametric motion examples:



$$\begin{pmatrix} u(\mathbf{x}) \\ v(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} a_2 & -a_3 \\ a_3 & a_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a_1 \\ a_4 \end{pmatrix}$$

rotation + scale + translation

Parametric motion examples:

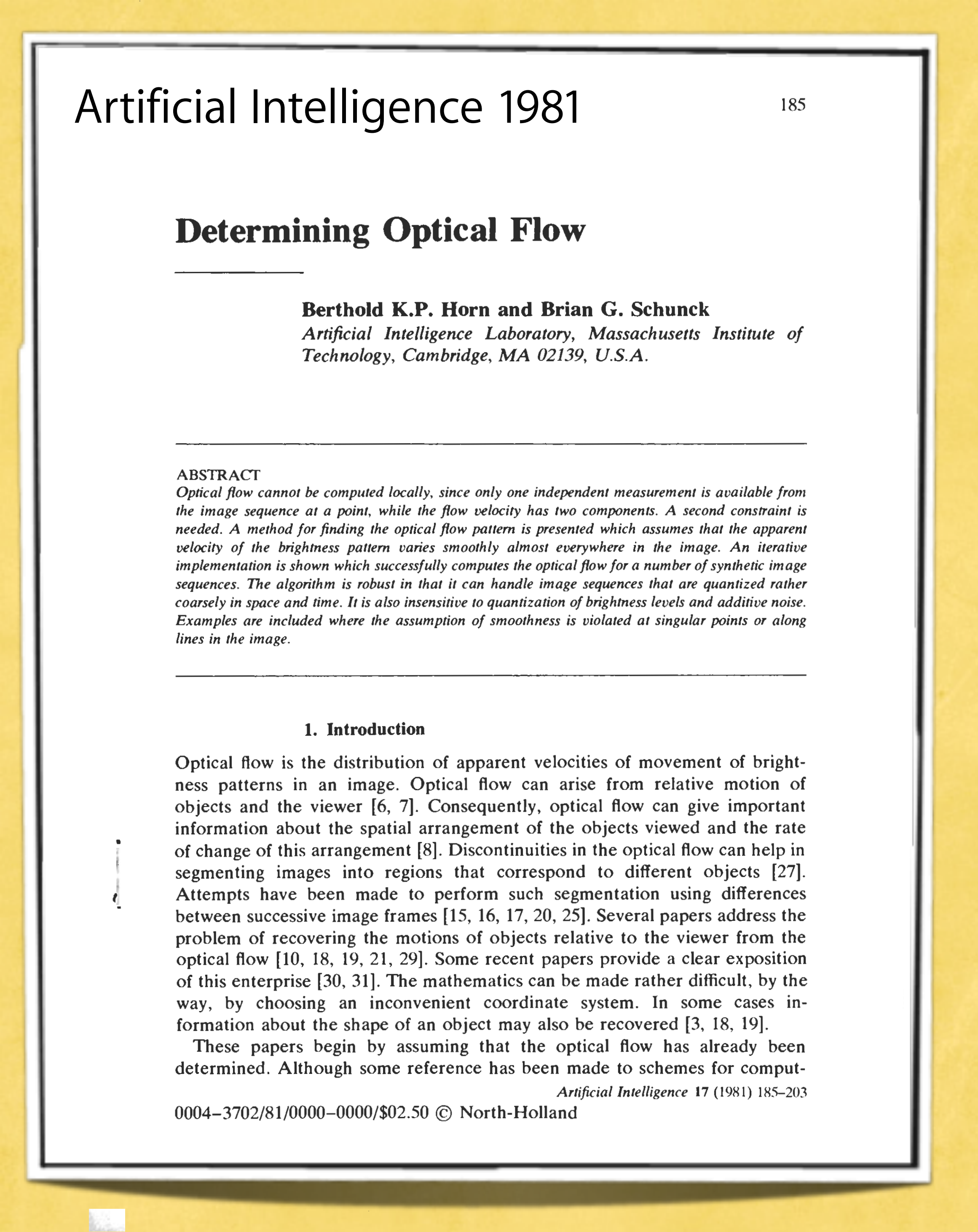


$$\begin{pmatrix} u(\mathbf{x}) \\ v(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} a_2 & a_3 \\ a_5 & a_6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a_1 \\ a_4 \end{pmatrix}$$

rotation + scale + shear
+ translation

Global

Global: Horn-Schunck



Artificial Intelligence 1981

Determining Optical Flow

Berthold K.P. Horn and Brian G. Schunck
Artificial Intelligence Laboratory, Massachusetts Institute of Technology, Cambridge, MA 02139, U.S.A.

ABSTRACT
Optical flow cannot be computed locally, since only one independent measurement is available from the image sequence at a point, while the flow velocity has two components. A second constraint is needed. A method for finding the optical flow pattern is presented which assumes that the apparent velocity of the brightness pattern varies smoothly almost everywhere in the image. An iterative implementation is shown which successfully computes the optical flow for a number of synthetic image sequences. The algorithm is robust in that it can handle image sequences that are quantized rather coarsely in space and time. It is also insensitive to quantization of brightness levels and additive noise. Examples are included where the assumption of smoothness is violated at singular points or along lines in the image.

1. Introduction

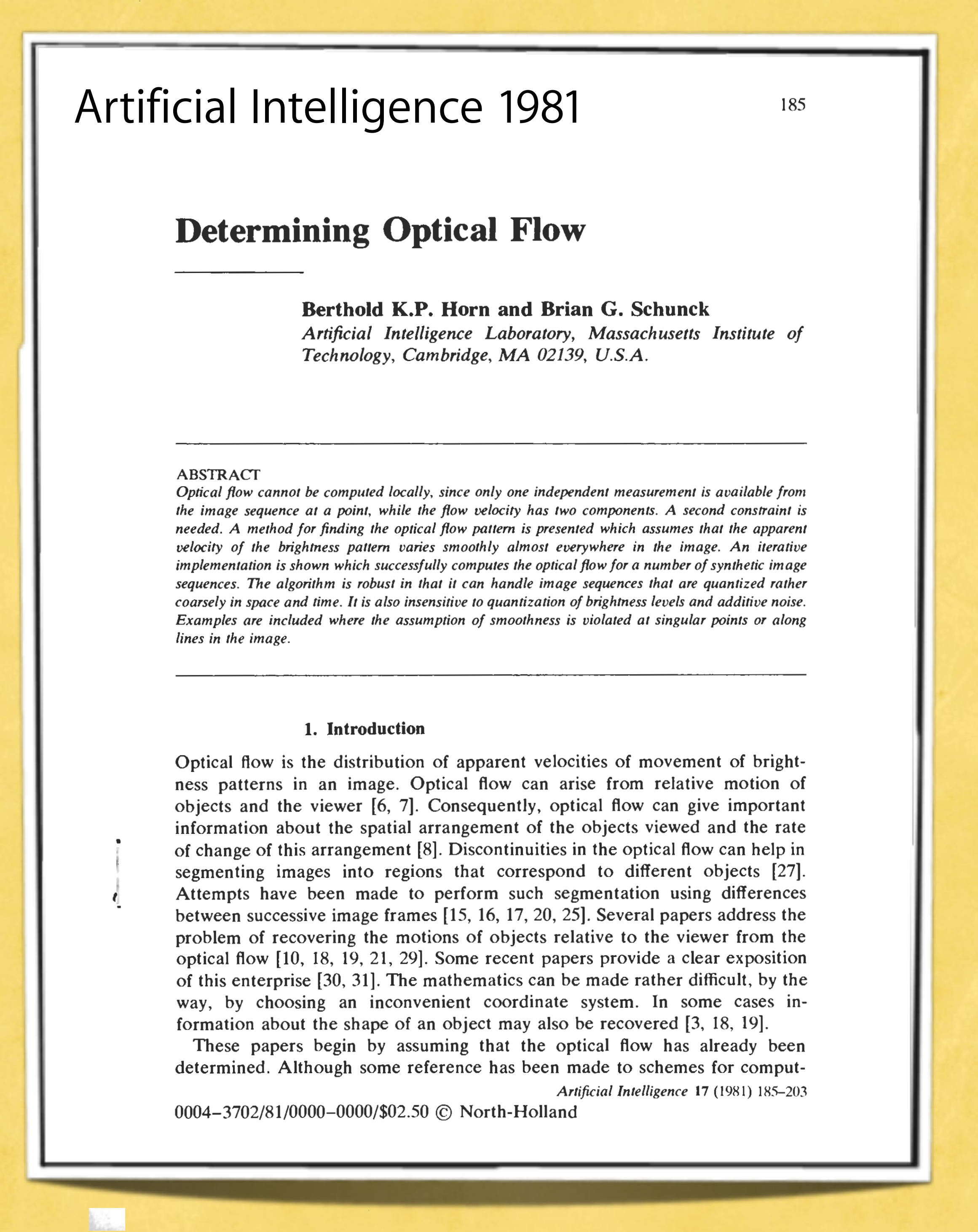
Optical flow is the distribution of apparent velocities of movement of brightness patterns in an image. Optical flow can arise from relative motion of objects and the viewer [6, 7]. Consequently, optical flow can give important information about the spatial arrangement of the objects viewed and the rate of change of this arrangement [8]. Discontinuities in the optical flow can help in segmenting images into regions that correspond to different objects [27]. Attempts have been made to perform such segmentation using differences between successive image frames [15, 16, 17, 20, 25]. Several papers address the problem of recovering the motions of objects relative to the viewer from the optical flow [10, 18, 19, 21, 29]. Some recent papers provide a clear exposition of this enterprise [30, 31]. The mathematics can be made rather difficult, by the way, by choosing an inconvenient coordinate system. In some cases information about the shape of an object may also be recovered [3, 18, 19].

These papers begin by assuming that the optical flow has already been determined. Although some reference has been made to schemes for comput-

Artificial Intelligence **17** (1981) 185–203
0004–3702/81/0000–0000/$02.50 © North-Holland

Global: Horn-Schunck

6816 citations !!!



Artificial Intelligence 1981 185

Determining Optical Flow

Berthold K.P. Horn and Brian G. Schunck
Artificial Intelligence Laboratory, Massachusetts Institute of Technology, Cambridge, MA 02139, U.S.A.

ABSTRACT
Optical flow cannot be computed locally, since only one independent measurement is available from the image sequence at a point, while the flow velocity has two components. A second constraint is needed. A method for finding the optical flow pattern is presented which assumes that the apparent velocity of the brightness pattern varies smoothly almost everywhere in the image. An iterative implementation is shown which successfully computes the optical flow for a number of synthetic image sequences. The algorithm is robust in that it can handle image sequences that are quantized rather coarsely in space and time. It is also insensitive to quantization of brightness levels and additive noise. Examples are included where the assumption of smoothness is violated at singular points or along lines in the image.

1. Introduction

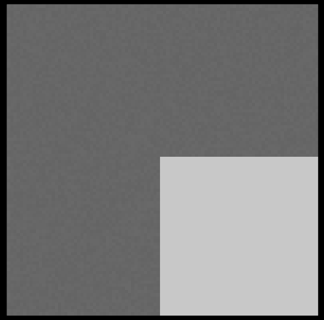
Optical flow is the distribution of apparent velocities of movement of brightness patterns in an image. Optical flow can arise from relative motion of objects and the viewer [6, 7]. Consequently, optical flow can give important information about the spatial arrangement of the objects viewed and the rate of change of this arrangement [8]. Discontinuities in the optical flow can help in segmenting images into regions that correspond to different objects [27]. Attempts have been made to perform such segmentation using differences between successive image frames [15, 16, 17, 20, 25]. Several papers address the problem of recovering the motions of objects relative to the viewer from the optical flow [10, 18, 19, 21, 29]. Some recent papers provide a clear exposition of this enterprise [30, 31]. The mathematics can be made rather difficult, by the way, by choosing an inconvenient coordinate system. In some cases information about the shape of an object may also be recovered [3, 18, 19].

These papers begin by assuming that the optical flow has already been determined. Although some reference has been made to schemes for comput-

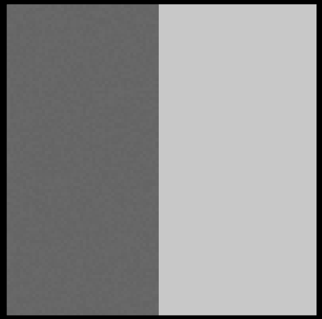
Artificial Intelligence **17** (1981) 185–203
0004–3702/81/0000–0000/$02.50 © North-Holland

Sketch: Horn-Schunck Flow

Local motion is inherently ambiguous



No ambiguity



Definite along the normal, ambiguous along the tangent

Totally ambiguous

Sketch: Horn-Schunck Flow

Horn and Schunck's Solution:

In addition to brightness constancy, impose spatial smoothness to the flow field.

$$\boxed{\arg\min_{u,v} \int\int (I_x u + I_y v + I_t)^2 + \alpha(\|\nabla u\|^2 + \|\nabla v\|^2) dx dy}$$

Sketch: Horn-Schunck Flow

Horn and Schunck's Solution:

In addition to brightness constancy, impose spatial smoothness to the flow field.

$$\boxed{\arg\min_{u,v} \int\int \underbrace{(I_x u + I_y v + I_t)^2}_{\uparrow \atop \textit{data term}} + \alpha(\|\nabla u\|^2 + \|\nabla v\|^2) dx dy}$$

Sketch: Horn-Schunck Flow

Horn and Schunck's Solution:

In addition to brightness constancy, impose spatial smoothness to the flow field.

$$\boxed{\arg\min_{u,v} \int\int \underbrace{(I_x u + I_y v + I_t)^2}_{\uparrow\ \text{data term}} + \alpha \underbrace{(\|\nabla u\|^2 + \|\nabla v\|^2)}_{\uparrow\ \text{smoothness term}} dxdy}$$

$$\|\nabla u\|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = u_x^2 + u_y^2$$

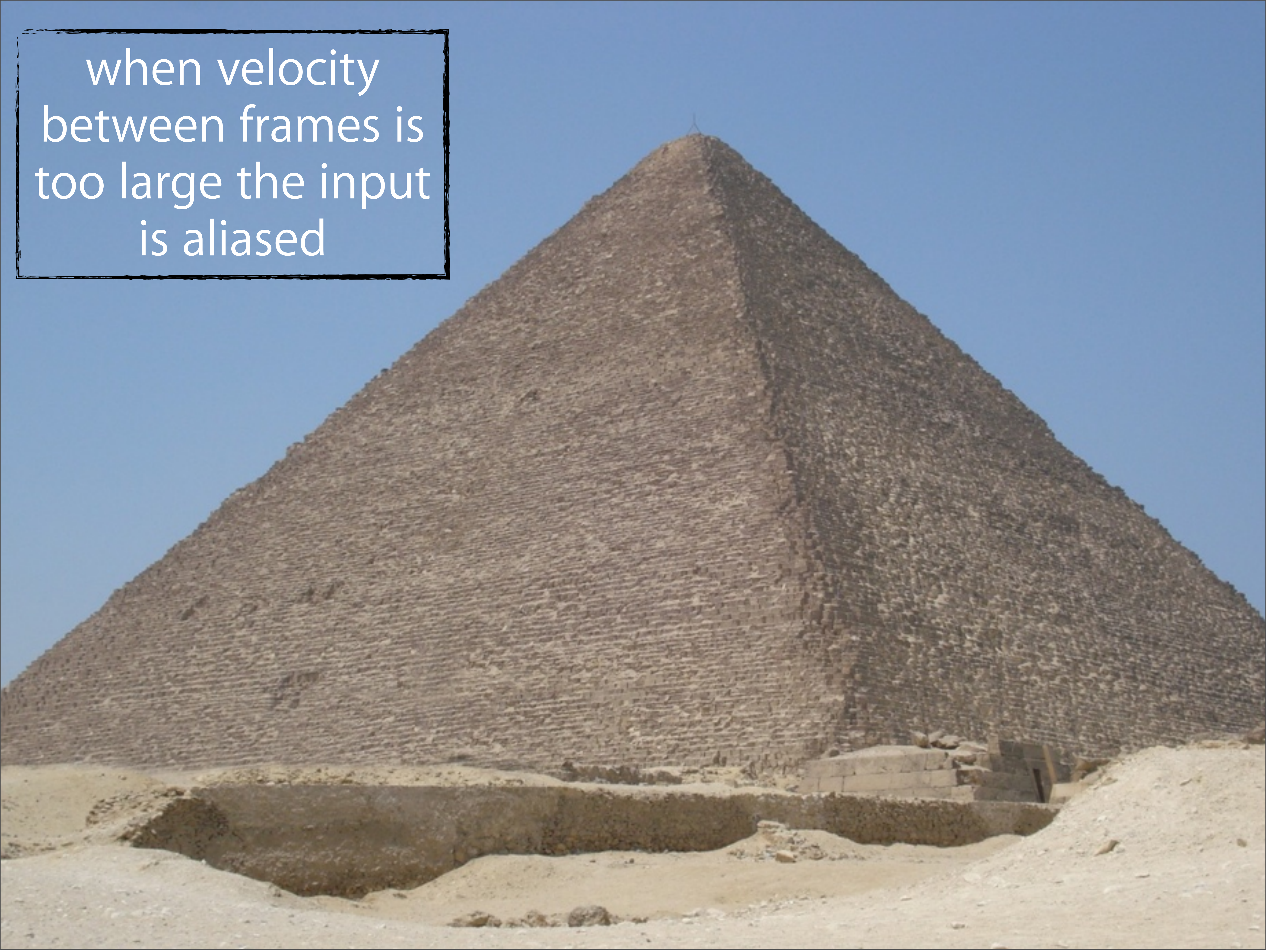
Sketch: Horn-Schunck Flow

Horn and Schunck's Solution:

In addition to brightness constancy, impose spatial smoothness to the flow field.

$$\boxed{\arg\min_{u,v} \int\int \underset{\substack{\uparrow \\ \textit{data term}}}{\underline{(I_x u + I_y v + I_t)^2}} + \overset{\substack{\textit{smoothness coefficient} \\ \downarrow}}{\alpha} \underset{\substack{\uparrow \\ \textit{smoothness term}}}{\underline{(\|\nabla u\|^2 + \|\nabla v\|^2)}} dx dy}$$

$$\|\nabla u\|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = u_x^2 + u_y^2$$



when velocity between frames is too large the input is aliased

Coarse-to-fine refinement

smooth & downsample

$I(x, y, t)$

$I(x, y, t+1)$

Coarse-to-fine refinement

$I(x, y, t)$

$I(x, y, t + 1)$

Coarse-to-fine refinement

$I(x, y, t)$

$I(x, y, t + 1)$

Coarse-to-fine refinement

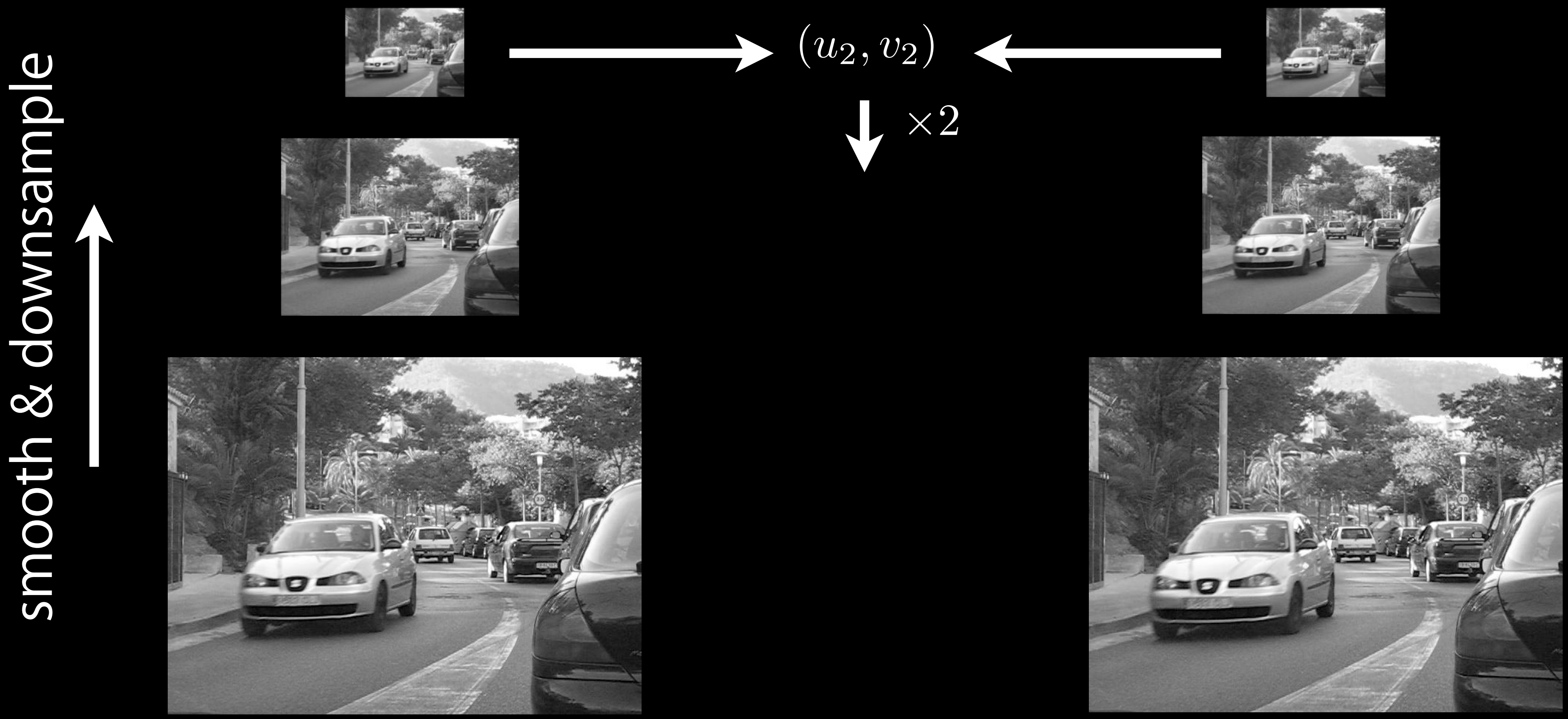
smooth & downsample

(u_2, v_2)

$I(x, y, t)$

$I(x, y, t+1)$

Coarse-to-fine refinement



smooth & downsample

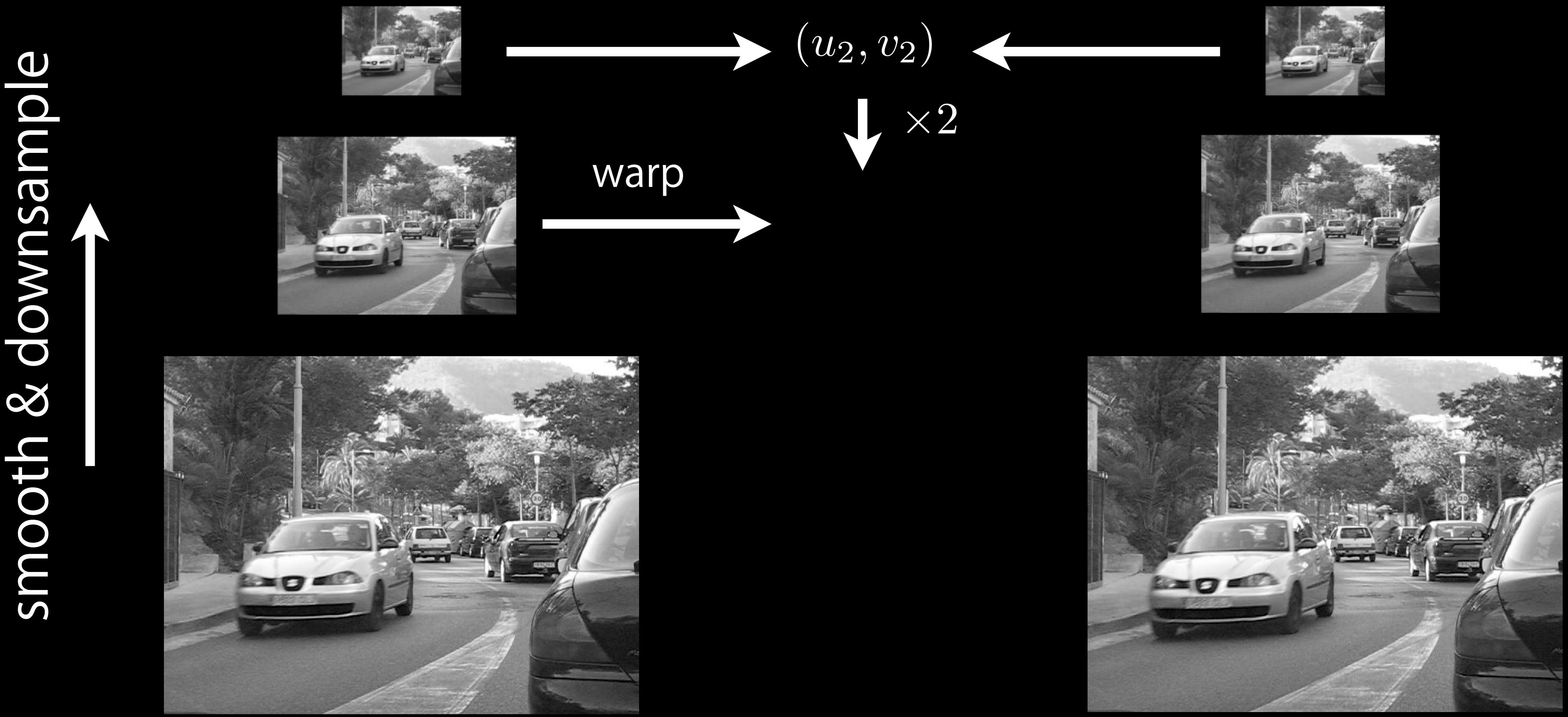
(u_2, v_2)

$\times 2$

$I(x, y, t)$

$I(x, y, t+1)$

Coarse-to-fine refinement



smooth & downsample

(u_2, v_2)

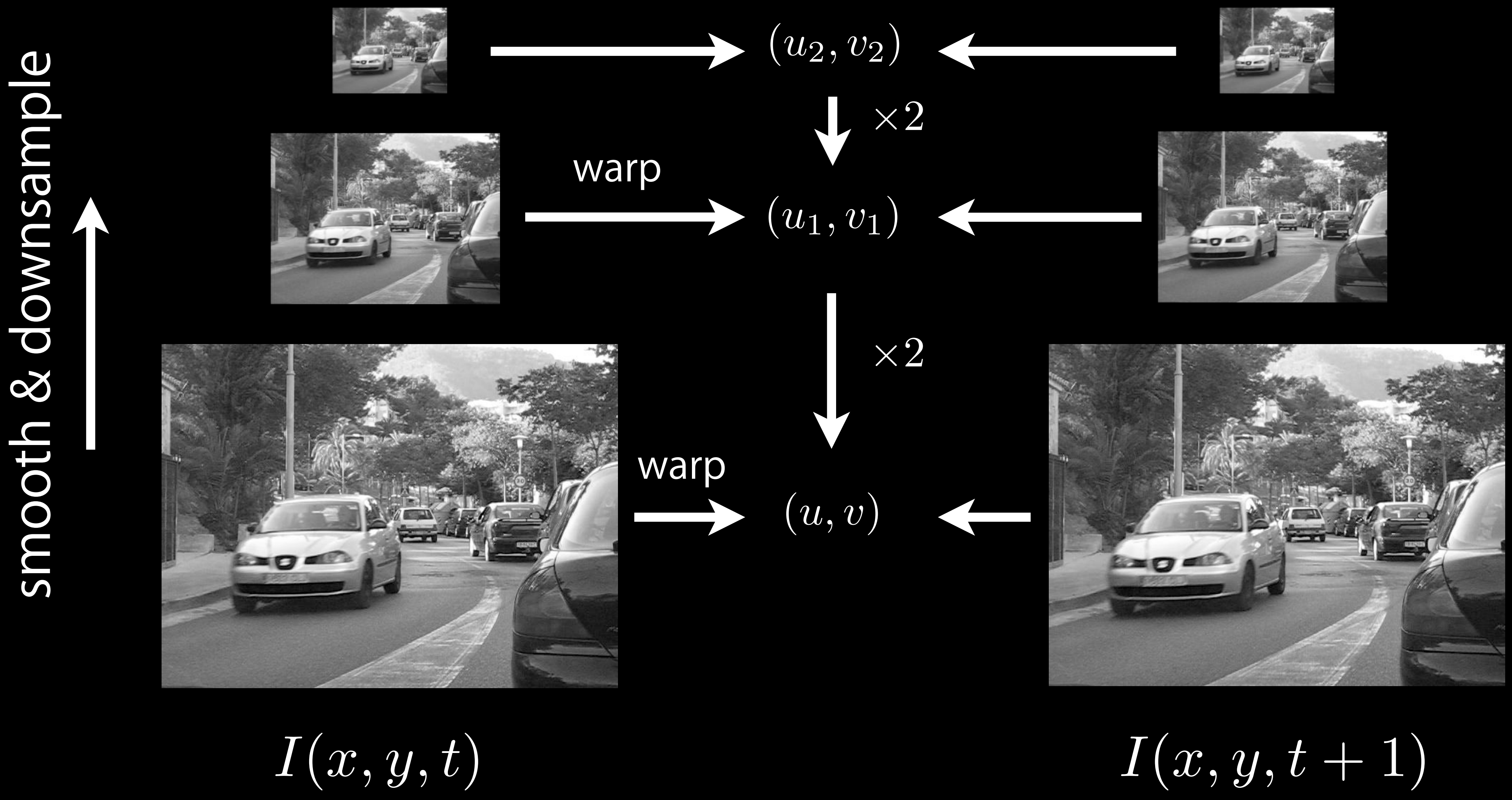
$\times 2$

warp

$I(x, y, t)$

$I(x, y, t + 1)$

Coarse-to-fine refinement



smooth & downsample

(u_2, v_2)

$\times 2$

warp

(u_1, v_1)

$\times 2$

warp

(u, v)

$I(x, y, t)$

$I(x, y, t+1)$