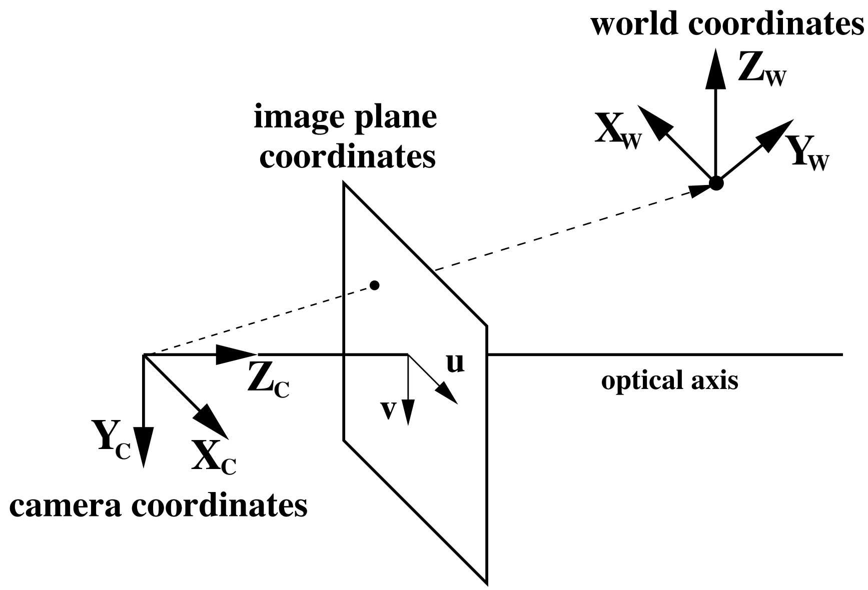
Camera model



world coordinate system coordinates (X_w, Y_w, Z_w),

camera coordinate system coordinates (X_c, Y_c, Z_c).

image homogeneous coordinates (u, v). The optical axis is the z-axis and the image plane is perpendicular to the optical axis. Intersection of the image plane with the optical axis is the point (u_o, v_o).

f is the distance of the image plane from the origin (effective focal length f in pixels).

$$u - u_o = f\frac{X_c}{Z_c} \quad v - v_o = f\frac{Y_c}{Z_c}.$$

We use s_x and s_y instead of f to capture different sizes of each pixel in mm (and not only): $s_x = f/d_x, s_y = f/d_y$.

Everything in Euclidean geometry!

No representation for points at infinity (intersections of parallel lines).

Remember that points at infinity project at finite points on images.

Projective geometry helps us how to deal with them.

At the end of projective geometry we will have learnt how to project points at infinity but also how to do virtual billboarding !!

Definition 1. *Given a set X, a relation $R \subset X \times X$ is called* **equivalence** *(and denoted with $x \sim y$ for $(x, y) \in R$) if it is reflexive ($x \sim x$), symmetric ($x \sim y \Rightarrow y \sim x$), and transitive ($x \sim y,\ y \sim z\ \Rightarrow x \sim z$).*
An **equivalence class** *with representative $a \in X$ is the subset of all elements of X which are equivalent to a.*

Definition 2. *Two elements of $\mathbb{R}^3 \setminus \{(0,0,0)\}$ are* **projectively equivalent**

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} \sim \begin{pmatrix} u' \\ v' \\ w' \end{pmatrix} \quad \textit{if} \quad \exists \lambda \in \mathbb{R} \setminus \{0\} : \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \lambda \begin{pmatrix} u' \\ v' \\ w' \end{pmatrix}$$

Definition 3. *The* **projective plane** $\mathbb{P}^2$ *is the set of all projective equivalence classes of* $\mathbb{R}^3 \setminus \{(0,0,0)\}$.

Example: Points in $\mathbb{R}^3$ lying in the same line through the origin are projectively equivalent. A line through the origin is then an equivalence class. Its representative can be for example the pair of antipodal intersections with the unit sphere.

Injection of $\mathbb{R}^2$ in $\mathbb{P}^2$:

- A point $(x', y') \in \mathbb{R}^2$ is injected in $\mathbb{P}^2$ with the addition of the homogeneous coordinate 1: $(x', y', 1) \in \mathbb{P}^2$.
- Only the points (x, y, w) of $\mathbb{P}^2$ with $w \neq 0$ can be mapped to $\mathbb{R}^2$ as $(x/w, y/w)$. If projective equivalence can be visualized as lines through the origin in $\mathbb{R}^3$, then $\mathbb{R}^2$ is injected as the plane $w = 1$.

Points and Lines

The line l going through points p and q in $\mathbb{P}^2$ reads

$$l \sim p \times q.$$

Three points p, q, r are collinear iff $p^T(q \times r) = 0$.

The intersection p of lines l and m reads

$$p \sim l \times m.$$

Three lines l, m, n are concurrent iff $l^T(m \times n) = 0$.

Projective Transformation

Definition 4. *A* **projective transformation** *is any invertible matrix transformation* $\mathbb{P}^2 \rightarrow \mathbb{P}^2$.

- A projective transformation A maps p to $p' \sim Ap$. This means that $\det(A) \neq 0$ and that there exists $\lambda \neq 0$ such that $\lambda p' = Ap$. Observe that we will write either $p' \sim Ap$ or $\lambda p' = Ap$.

- If A maps a point to Ap, then A maps a line l to $l' = A^{-T}l$. This can be shown starting from the line equation $l^Tp = 0$ and replacing p with $A^{-1}p'$.

- A projective transformation λA is the same as A since they map to projectively equivalent points. Hence, we will be able to determine a projective transformation only up to a scale factor.

- A projective transformation preserves incidence.
- Three collinear points are mapped to three collinear points and three concurrent lines are mapped to three concurrent lines.
- The latter might involve the case that the point of intersection is mapped to a point at infinity.
- Because of the incidence preservation, projective transformations are also called **collineations**.

How many points suffice to determine a projective transformation?

Assume that a mapping A maps the three points $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ to the non-collinear points a, b and c. Since $a \sim \alpha a$, $b \sim \beta b$, and $c \sim \gamma c$ we can write:

$$\begin{pmatrix} a & b & c \end{pmatrix} \sim \begin{pmatrix} \alpha a & \beta b & \gamma c \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It is obvious that for each choice of $\alpha, \beta, \gamma \neq 0$ we can build a matrix A mapping the points $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ to a, b and c. Let us assume that the same A maps $(1,1,1)$ to the point d. Then, the following should hold:

$$\lambda d = \begin{pmatrix} \alpha a & \beta b & \gamma c \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

hence

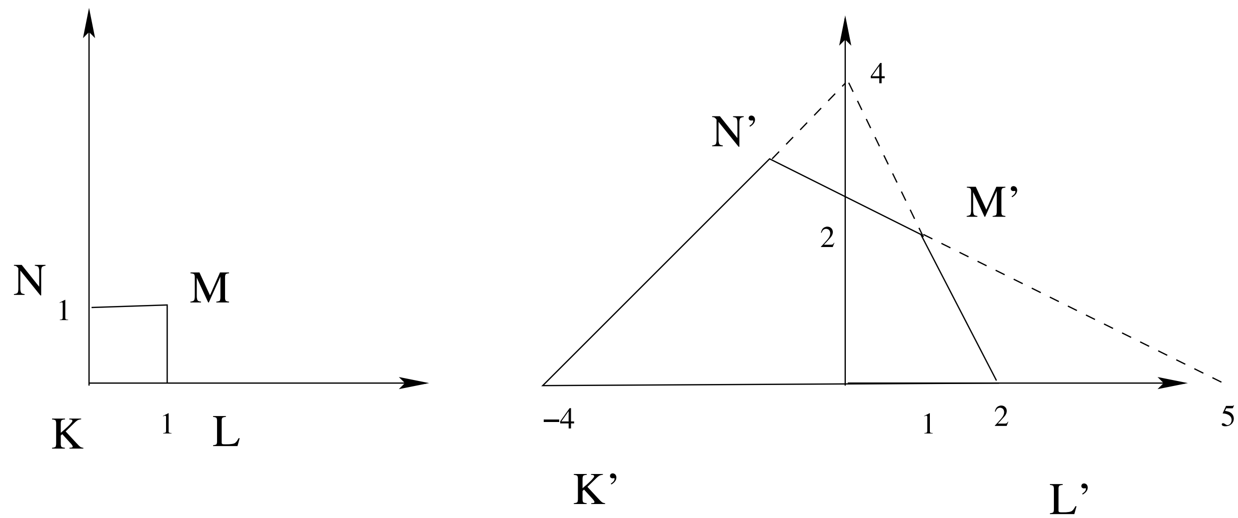
$$\lambda d = \alpha a + \beta b + \gamma c.$$

There always exist such $\lambda, \alpha, \beta, \gamma$ because four elements of $\mathbb{R}^3 \backslash \{(0,0,0)\}$ are always linearly dependent. Because a, b, c are not collinear, there exist unique $\alpha/\lambda, \beta/\lambda, \gamma/\lambda$ for writing this linear combination. Since A is the same as A/λ we solve for α, β, γ such that $d = \alpha a + \beta b + \gamma c$, which can be written as a linear system

$$\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = d.$$

Since a, b, c are not collinear we can always find a unique triple α, β, γ. The resulting projective transformation is $A = \begin{pmatrix} \alpha a & \beta b & \gamma c \end{pmatrix}$.

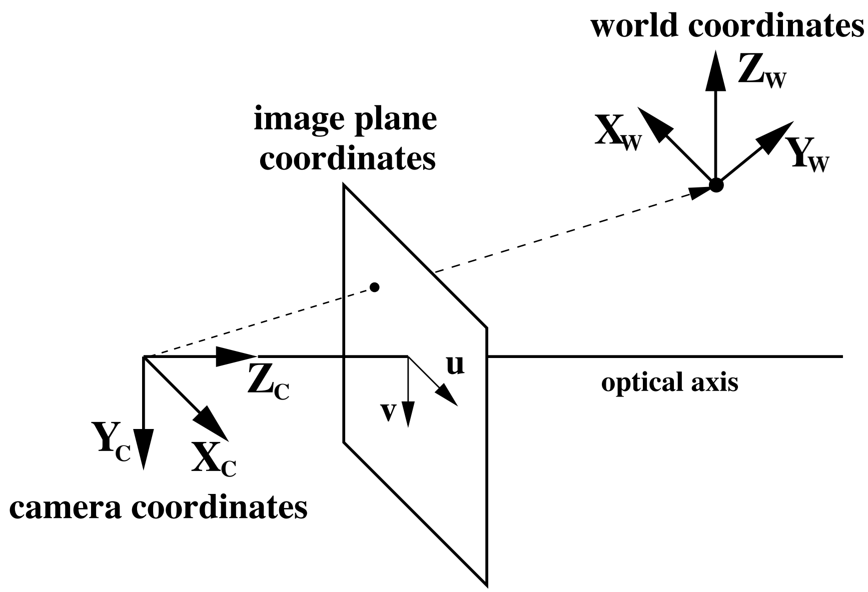
Proposition 1. *Four points not three of them collinear suffice to recover unambiguously a projective transformation.*



Find projective transformation mapping $(a, b, c, d) \rightarrow (a', b', c', d')$:

To determine this mapping we go through the four points used in the paragraph above. We find the mapping from $(1, 0, 0), etc$ to (a, b, c, d) and we call it T: $a \sim T(1, 0, 0)^T, etc$. We find the mapping from $(1, 0, 0), etc$ to (a', b', c', d') and we call it T': $a' \sim T'(1, 0, 0)^T, etc$. Then, back-substituing $(1, 0, 0)^T \sim T^{-1}a, etc$ we obtain that $a' = T'T^{-1}a, etc$. Thus, the required mapping is $T'T^{-1}$.

Camera models again



world coordinate system homogeneous coordinates $(X_w, Y_w, Z_w, W$

camera coordinate system homogeneous coordinates (X_c, Y_c, Z_c, W_c).

image homogeneous coordinates (u, v, w).

Projection of a point (X_c, Y_c, Z_c, W_c) on the image plane:

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} \sim \begin{pmatrix} s_x & 0 & u_o \\ 0 & s_y & v_o \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} X_c \\ Y_c \\ Z_c \\ W_c \end{pmatrix}.$$

Transformation from camera to world reads as follows:

$$\begin{pmatrix} X_c \\ Y_c \\ Z_c \\ W_c \end{pmatrix} = \begin{pmatrix} R & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X_w \\ Y_w \\ Z_w \\ W_w \end{pmatrix}.$$

Projection from world to the image plane is a singular tranformation from $\mathbb{P}^3$ to $\mathbb{P}^2$:

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} \sim \begin{pmatrix} s_x & 0 & u_o \\ 0 & s_y & v_o \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} R & t \end{pmatrix} \begin{pmatrix} X_w \\ Y_w \\ Z_w \\ W_w \end{pmatrix} = \begin{pmatrix} \tilde{M} & \tilde{m} \end{pmatrix} \begin{pmatrix} X_w \\ Y_w \\ Z_w \\ W_w \end{pmatrix}$$

If $W_w \neq 0$, a point is not at infinity, the projection equation can be written as

$$\frac{\lambda}{W_w} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \tilde{M} \begin{pmatrix} X_w/W_w \\ Y_w/W_w \\ Z_w/W_w \end{pmatrix} + \tilde{m}$$

or

$$\begin{pmatrix} X_w/W_w \\ Y_w/W_w \\ Z_w/W_w \end{pmatrix} = \lambda' \tilde{M}^{-1} \begin{pmatrix} u \\ v \\ w \end{pmatrix} - \tilde{M}^{-1}\tilde{m}.$$

Projection from a plane in the world to the camera is a projective transformation

Suppose that everything lies in the plane $Z_w = 0$, for example that we place a world coordinate system with xy-plane coincident with the ground plane of Walnut Street. Assume that the rotation matrix above has columns

$$R = \begin{pmatrix} r_1 & r_2 & r_3 \end{pmatrix}.$$

Setting $Z_w = 0$

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} \sim \begin{pmatrix} s_x & 0 & u_o \\ 0 & s_y & v_o \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} r_1 & r_2 & t \end{pmatrix} \begin{pmatrix} X_w \\ Y_w \\ W_w \end{pmatrix} = H \begin{pmatrix} X_w \\ Y_w \\ W_w \end{pmatrix}$$

a projective transformation from $\mathbb{P}^2$ to $\mathbb{P}^2$!

Its determinant reads

$$\det(H) = s_x s_y t^T (r_1 \times r_2).$$