<1/13 <Filtering>

$f(t) \longrightarrow \boxed{\text{system}} \longrightarrow g(t)$

$f: \mathbb{R} \to \mathbb{R}$ $\qquad$ $g: \mathbb{R} \to \mathbb{R}$

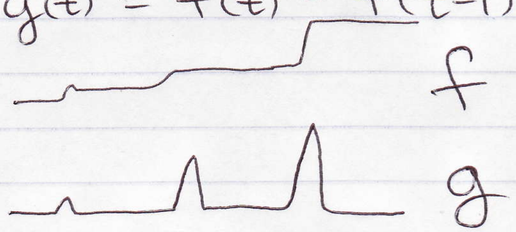
< Linear System >

$f_1(t) \longrightarrow \boxed{} \longrightarrow g_1(t)$

$f_2(t) \longrightarrow \boxed{} \longrightarrow g_2(t)$

$af_1(t) + bf_2(t) \rightarrow \boxed{} \rightarrow ag_1(t) + bg_2(t)$

example) $g(t) = f(t) - f(t-1)$



$g_1(t) = f_1(t) - f_1(t-1)$

$g_2(t) = f_2(t) - f_2(t-1)$

$ag_1(t) + bg_2(t) = af_1(t) + bf_2(t) - af_1(t-1) - bf_2(t-1)$

$= (af_1 + bf_2)(t) - (af_1 + bf_2)(t-1)$

Counter example)

$g(t) = \min\{f(t), f(t-1)\}$

Question. Is there a formula for such a system?

1/20 Filtering Coutinued

Szeliski 3.2 & 3.3

PART I : Image Processing (ch3)

PART II : Geometry (ch2)

* Linear System

$f(t) \longrightarrow \boxed{} \longrightarrow g(t)$

$f: \mathbb{R} \rightarrow \mathbb{R}$

$a_1 f_1(t) + a_2 f_2(t) \longrightarrow \boxed{} \longrightarrow a_1 g_1(t) + a_2 g_2(t)$

example) $g(t) = f(t) - f(t-1)$

Counter example) $g(t) = \max\{f(t), f(t-1)\}$

$g(t) = f^2(t)$

* Shift - invariant system

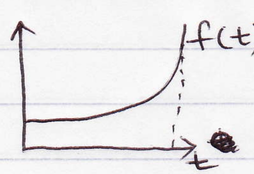
$f(t) \longrightarrow \boxed{} \longrightarrow g(t)$

then

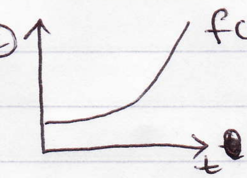
$f(t-t_0) \longrightarrow \boxed{} \longrightarrow g(t-t_0)$

example) $g(t) = f(t) - f(t-1)$

Counter example)

① 

$g(t) = t f(t)$

② 

(mean filter with variance kernel width)

$g(t) = \frac{1}{N} \sum_{i=-\frac{N}{2}}^{\frac{N}{2}} f(t+i)$ where $N = \sqrt{t}$

Question. Is there a formula for characterizing all linear shift-invariant system?

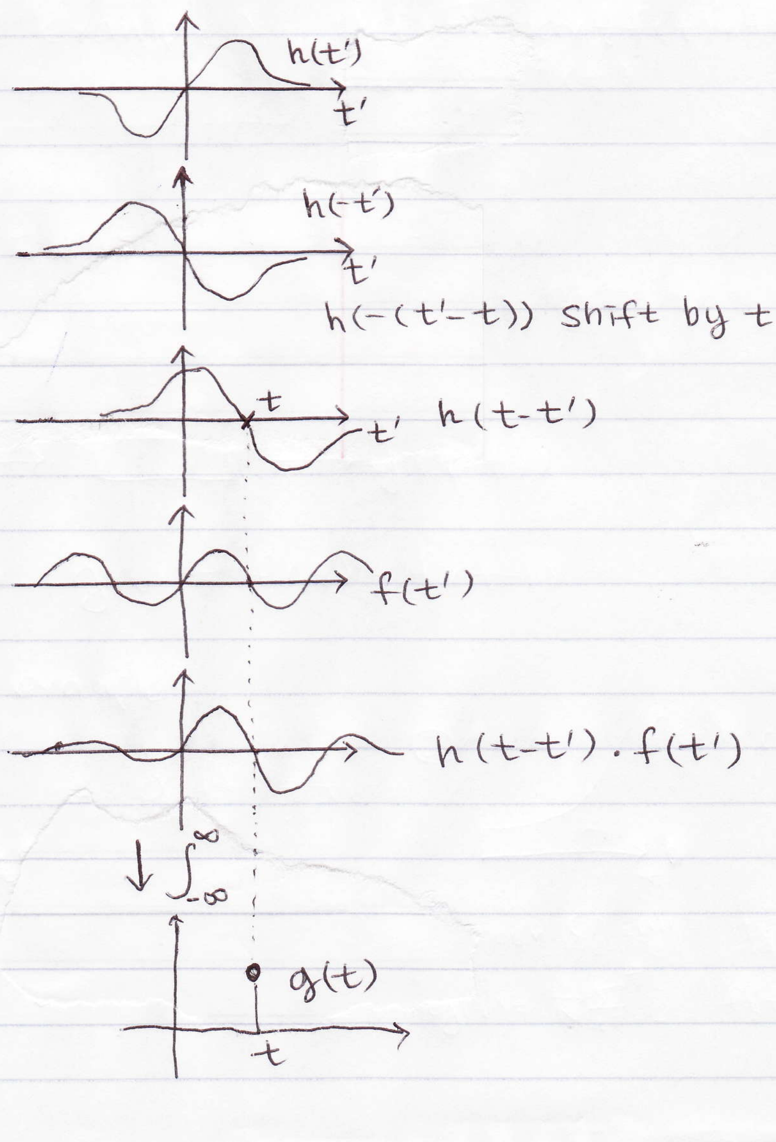
discrete intuition : $g(t) = \sum_i h_i f(t-i)$

continuous convolution: $g(t) = \int_{-\infty}^{\infty} h(t-t') f(t') dt'$

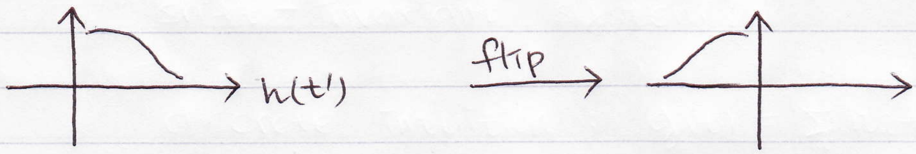
$f(t) \longrightarrow \boxed{h} \longrightarrow g(t)$

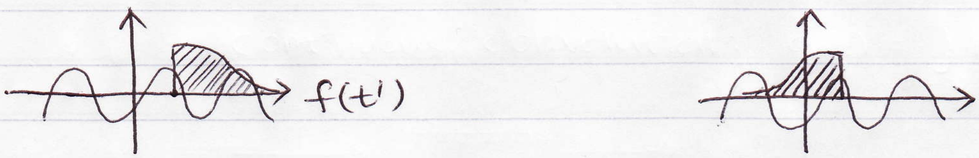
$h(t)$ impulse response

ex)



Why the reflection ?
(irrelevant if $h(-t') = h(t')$ )



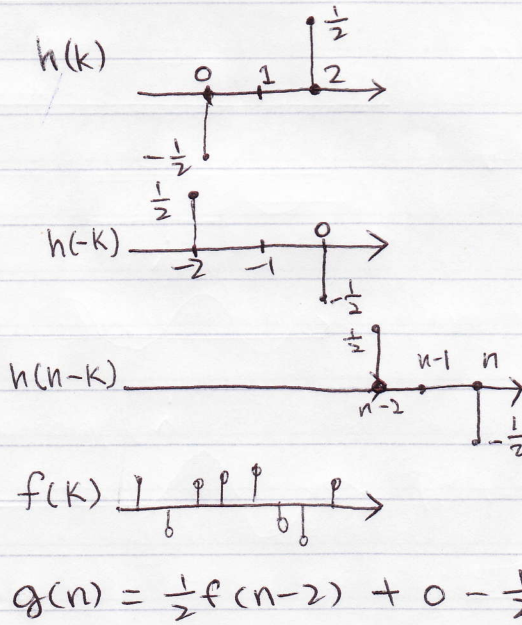


<u>causal system.</u> (causality, irrelevant to images, relevant to video)

if $f(t) = 0$ for $t < 0$
then $g(t) = 0$ for $t < 0$
$\Rightarrow$ Response at t cannot depend on $t' > t$

Parenthesis : <u>Discrete finite filters</u>

$$g(n) = \sum_{k=-N}^{0} h(n-k) f(k)$$



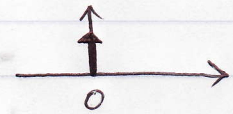
$h(n) = \alpha f(n) + (1-\alpha) h(n-1)$
recursive fuction
(ex: BG subtraction in video)

$g(n) = \frac{1}{2} f(n-2) + 0 - \frac{1}{2} f(n)$
Discrete Finite Impulse response (FIR)
(cf same with $\sum_{k=-\infty}^{\infty}$ $\rightarrow$ Discrete Infinite IR)

* Dirac Impulse $\delta(t)$

Informal Definition : $\delta(t) = 0$ for $t \neq 0$

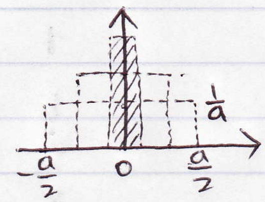
and $\int_{-\infty}^{\infty} \delta(t)\, dt = 1$



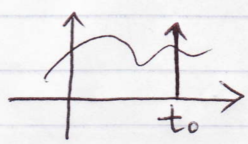
Incorrect : $\delta(0) = 1$

Other definition : $\delta(t) = \lim\limits_{a \to 0} \Pi_a(t)$

$$\Pi_a(t) = \begin{cases} \frac{1}{a} & |t| \leq \frac{a}{2} \\ 0 & \text{elsewhere} \end{cases}$$



Absorption property : $\int_{-\infty}^{\infty} f(t)\, \delta(t - t_0)\, dt = f(t_0)$



Can be written $\int f(t)\, \delta(t_0 - t)\, dt$ (since $\delta(t - t_0) = \delta(t_0 - t)$)

: Convolution

$\delta(t) \rightarrow \boxed{h(t)} \longrightarrow h(t)$

Impulse response

I exechanged the roles of input and filter.

$$\int f(t')\, h(t - t')\, dt' = \int f(t - t')\, h(t')\, dt'$$

* **harmonic**

$$\underset{\text{real}}{\cos(\omega t)} + j\,\underset{\text{imaginary}}{\sin(\omega t)} = e^{j\omega t} \qquad e^{j\pi} = -1$$

$i^2 = -1$

$j = -1$

$$e^{j\omega t} \rightarrow \boxed{h(t)} \xrightarrow{g(t)} = \int_{-\infty}^{\infty} h(t')\, e^{j\omega(t-t')}\, dt'$$

$$= e^{j\omega t} \underbrace{\int_{-\infty}^{\infty} h(t')\, e^{-j\omega t'}\, dt'}_{H(\omega)}$$

matrix theory
$Ax = \lambda x$
$x \rightarrow \boxed{A} \rightarrow Ax$
X are eigen vectors of A

$$e^{j\omega t} \rightarrow \boxed{} \rightarrow H(\omega)\, e^{j\omega t}$$

∴ $e^{j\omega t}$ is an eigenfuction of any linear ~~of~~ shift-invariant system $h(t)$ with eigen value $H(\omega) = \int_{-\infty}^{\infty} h(t)\, e^{-j\omega t}\, dt$

Note: Any discrete linear system
Can be written in matrix-vector form $x \rightarrow Ax$
If the system is also shift-invariant then A has a special form (convolution of Toeplitz Matrix).

* Fourier Transform

*negated

$$\mathcal{F}\{h(t)\} = H(\omega) = \int_{-\infty}^{\infty} h(t)\, e^{-j\omega t}\, dt$$

$$\mathcal{F}\{H(\omega)\} = h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega)\, e^{j\omega t}\, d\omega \qquad \omega = 2\pi s \rightarrow \frac{1}{\text{sec}} \text{ or } \frac{1}{\text{pixels}}$$

be careful on definition

Reading 3.2, 3.3
Castleman 9, 10