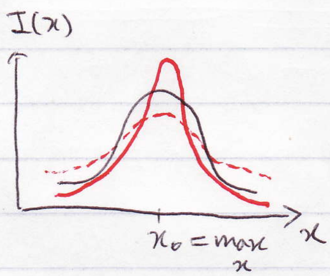
02/17 Features Continued.

- detection ⇒ Scale and rotation invariant
- description

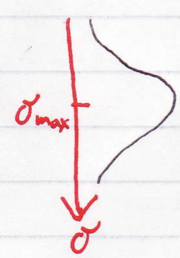
* Detection of intrinsic scale.
This used to normalize any neighborhood to a fixed size (e.g 16x16)

$\boxed{\text{Scale space}}$

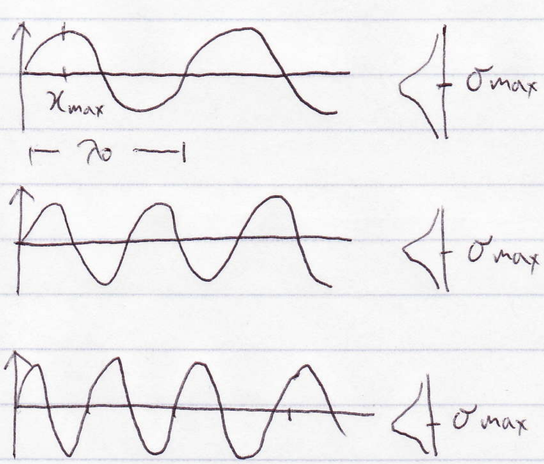
Intuitively :



Template(σ) * I(x)



<u>Example of sinusoid</u> : Build a template(σ) which will give a maximum along scale which is $\underline{\underline{\sigma_{max} \sim \lambda_0}}$



$$x_{max} = \underset{x}{\operatorname{argmax}} f(x)$$

$$f(x) = \sin w_0 x \qquad w_0 = \frac{2\pi}{\lambda_0}$$

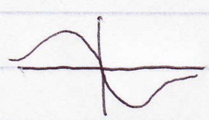
$$f(x) = \sin(w_0 x) \qquad w_0 = \frac{2\pi}{\lambda_0}$$

$$g(x,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} \qquad \text{(gaussian)}$$

$$g(x,\sigma) * f(x) = e^{-\frac{w_0^2\sigma^2}{2}} \sin w_0 x \qquad \left(\text{at } x_{max} \ e^{\frac{-w_0^2\sigma^2}{2}} \text{ maximized at } \sigma = 0\right)$$

Note: $\frac{d}{dx} f(x) \quad \circ\!\!-\!\!\bullet \quad jwF(w)$

$$\frac{d}{dx} g(x,\sigma) * f(x) = \underbrace{w_0 e^{\frac{-w_0^2\sigma^2}{2}}}_{\text{still maximized for } \sigma=0} \cos w_0 x$$



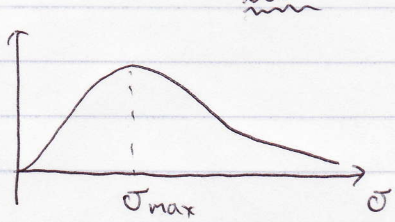
Amplitude of m-th derivative $w_0^m e^{-\frac{w_0^2\sigma^2}{2}}$

Amplitude decreases with scale!

How can we modify Gaussian derivatives to achieve $\sigma_{max} \sim \lambda_0 \sim \frac{1}{w_0}$

Normalized Gaussian derivative which yields an amplitude

$$\sigma^m w_0^m e^{\frac{-w_0^2\sigma^2}{2}}$$



$$\frac{\partial}{\partial\sigma} = m\sigma^{m-1} w_0^m e^{\frac{-w_0^2\sigma^2}{2}} - w_0^2\sigma e^{-\frac{w_0^2\sigma^2}{2}} \sigma^m w_0^m$$

$$= e^{-\frac{w_0^2\sigma^2}{2}} w_0^m (m\sigma^{m-1} - \sigma^{m+1} w_0^2)$$

$$= 0$$

$$\Rightarrow m\sigma^{-1} = w_0^2\sigma$$

$$\Rightarrow \boxed{\sigma^2 w_0^2 = m}$$

$$\therefore \sigma_{max} = \sqrt{m}\frac{1}{w_0}$$

$$= \sqrt{m}\frac{\lambda_0}{2\pi}$$

How do we obtain this normalized derivative?

$$\xi = \frac{x}{\sigma}$$

normalized derivative $\frac{\partial}{\partial \xi} = \frac{\partial g}{\partial x} \cdot \frac{\partial x}{\partial \xi} = \partial \frac{\partial g}{\partial x}$

Why Gaussian?

$$g(x, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-x^2}{2\sigma^2}}$$

$$\frac{\partial g(x,\sigma)}{\partial \sigma} = \frac{1}{\sqrt{2\pi}}\left(-\frac{1}{\sigma^2} e^{\frac{-x^2}{2\sigma^2}} + \frac{1}{\sigma}\left(-\frac{x^2}{2}\right)\frac{-2}{\sigma^3} e^{\frac{-x^2}{2\sigma^2}}\right)$$

$$= \underline{\frac{1}{\sigma^2\sqrt{2\pi}}\left(e^{\frac{-x^2}{2\sigma^2}}\right)\left(-1 + \frac{x^2}{\sigma^2}\right)}$$

$$\frac{\partial g(x,\sigma)}{\partial x} = \frac{1}{\sigma\sqrt{2\pi}}\left(\frac{-\cancel{2}x}{\cancel{2}\sigma^2} e^{\frac{-x^2}{2\sigma^2}}\right)$$

$$\frac{\partial^2 g(x,\sigma)}{\partial x^2} = \frac{1}{\sigma^3\sqrt{2\pi}}\left(-e^{\frac{x^2}{2\sigma^2}} - x\frac{-2x}{2\sigma^2} e^{\frac{-x^2}{2\sigma^2}}\right) = \underline{\frac{e^{\frac{-x^2}{2\sigma^2}}}{\sigma^3\sqrt{2\pi}}\left(-1 + \frac{x^2}{\sigma^2}\right)}$$

$$\therefore \frac{\partial g(x,\sigma)}{\partial \sigma} = \sigma \frac{\partial g^2(x,\sigma)}{\partial x^2} \quad \text{let } t = \sigma^2 \quad \boxed{\frac{\partial g}{\partial t} = \frac{1}{2}\frac{\partial^2 g}{\partial^2 x}}$$

diffusion equation

PDE theory : $\frac{\partial L}{\partial \sigma} = \sigma \frac{\partial^2 L}{\partial x^2}$

boundary condition $L(x, \sigma=0) = f(x)$

image

Solution to this equation is

$$\boxed{L(x,\sigma) = \text{gaussian}(x,\sigma) * f(x)}$$

template   scale space

Intrinsic scale $\sigma_{max} = \arg\max_{\sigma} \left(\arg\max_{x} L^{m}(x, \sigma) \right)$ where L^m m-th derivative.

Edge detection $m=1$

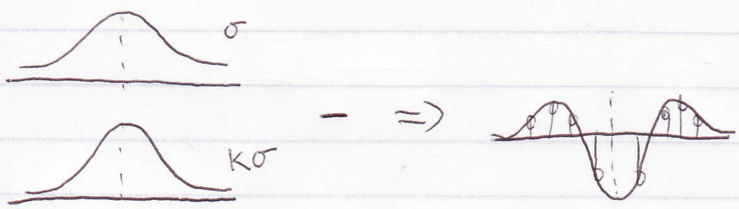
$$\frac{dg}{dx} * f(x)$$

SIFT (blob) $\frac{d^2 g}{dx^2} * f(x)$

$$2D: \left(\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} \right) * f(x, y)$$

Discrete case 2^{nd} derivative

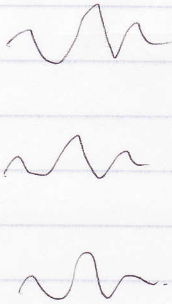
$$\frac{\partial g}{\partial \sigma} = \sigma \frac{\partial^2 g}{\partial x^2}$$



$$\frac{\partial g}{\partial \sigma} \approx \frac{(g(x, k\sigma) * f(x)) - (g(x, \sigma) * f(x))}{k\sigma - \sigma}$$

$\Rightarrow$ DOG $\approx$ LOG

In Practice



$g(x, \sigma) * f(x)$

$g(x, k\sigma) * f(x)$

$g(x, 2k\sigma) * f(x)$

minus

$$\frac{\ominus}{\sigma(k-1)} = \frac{\partial^2 g}{\partial x^2} \sigma$$

normalised $\frac{\partial}{\partial \xi} = \sigma^2 \frac{\partial^2 g}{\partial x^2}$

$\hookrightarrow$ Just $\ominus$, without division.

Why is σ sampled in octaves?

$$\frac{\partial g}{\partial \sigma} \approx \frac{g(\sigma+\Delta\sigma) - g(\sigma)}{\Delta\sigma}$$

$$\frac{g(k\sigma) - g(\sigma)}{k\sigma - \sigma}$$

Subsequent convolution

$$g(x, \sigma_2) * \left(g(x, \sigma_1) * f(x) \right)$$

$$= g(x, \sqrt{\sigma_1^2 + \sigma_2^2}) * f(x)$$

$\because$ Fourier $e^{-\frac{w_0^2 \sigma_1^2}{2}} e^{-\frac{w_0^2 \sigma_2^2}{2}} F(w)$

$$= e^{-\frac{w_0^2 (\sqrt{\sigma_1^2 + \sigma_2^2})^2}{2}} F(w)$$

$\sigma_1 = \sigma_2$ same as convolution with $\sqrt{2}\sigma$

then with $\sqrt{2}\sigma$ $\quad 2\sigma$

2σ $\quad 2\sqrt{2}\sigma$