Construction of Manifolds and Parametric Pseudo-Manifolds from Gluing Data

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ABSTRACT

This paper presents the theoretical framework of a new constructive solution for the problem of fitting a smooth surface to a given triangle mesh. Our construction is based on the manifold-based approach pioneered by Grimm and Hughes. The key idea behind this approach is to define a surface by overlapping surface patches via a gluing process, as opposed to stitching them together along their common boundary curves.

Our new manifold-based solution possesses most of the best features of previous constructions. In particular, our construction is simple, compact, powerful, and flexible in ways of defining the geometry of the resulting surface. Unlike some of the most recent manifold-based solutions, ours has been devised to work with triangle meshes. These meshes are far more popular than any other kind of mesh encountered in computer graphics and geometry processing applications. This paper provides a mathematically sound theoretical framework for our method, using what we call *sets of gluing data*. This theoretical framework slightly improves upon the one given by Grimm and Hughes, which was used by most manifold-based constructions introduced before.

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Chapter 1

Introduction

Fitting a surface with guaranteed topology and continuity to the vertices of a mesh (triangle or quadrilateral) of arbitrary topology has been a topic of major research interest for many years. This is mainly due to the fact that, in general, meshes of arbitrary topology cannot be parametrized on a single rectangular domain and have no restriction on vertex connectivity. Much of the previous research efforts has been focused on stitching parametric polynomial patches together along their seams (see Figure 1.1).

Each patch is the image of a distinct parametrization of a closed, planar domain. Because the patches need to be "pieced" together, there are natural smoothness concerns along the borders where they join. It turns out that ensuring continuity along the borders has proved to be a difficult problem, in particular for closed[1] meshes.

Although there is a large number of C^k constructions, where k is a finite integer, based on the "stitching" paradigm and catered to triangle meshes[2] (see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16]), only a few go beyond C^2-continuity (i.e., [9, 11, 16]). However, higher order constructions suffer from the following drawbacks:

- **High order polynomial patches**. To enforce high order continuity, high order polynomial patches, whose degree rapidly grows with the desired degree, k, of continuity, are required. A recent exception is the construction in [16], which is capable of producing G^k-continuous surfaces of low degree.
- **Free parameters**. The geometry of the polynomial patches is defined by a finite amount of points, called *control points*, whose locations are determined by *free parameters* of the

[1]Meshes without boundary, or equivalently, in which each edge is shared by exactly two triangles (or quadrilaterals).

[2]Some of them are actually G^k-continuous, which is a measure of continuity that subsumes strict parametric continuity.

construction. Free parameters can be used to adjust and fair the shape of the patches. However, an automatic procedure for optimizing these parameters is rarely found among the majority of the constructions. As a result, shape tuning is up to the designer and it can become an extremely laborious task if the triangle mesh has a large number of triangles (as the number of patches is in general no smaller than the number of triangles).

- **Lack of shape control**. Continuity is ensured by maintaining constraints on the position of the control points, which limits the freedom to move those points freely to achieve a desirable shape.
- **Lack of simplicity**. Higher order constructions are in general complex. Very few of them were ever implemented, and the visual quality of their resulting surfaces was typically inferior to lower degree constructions.

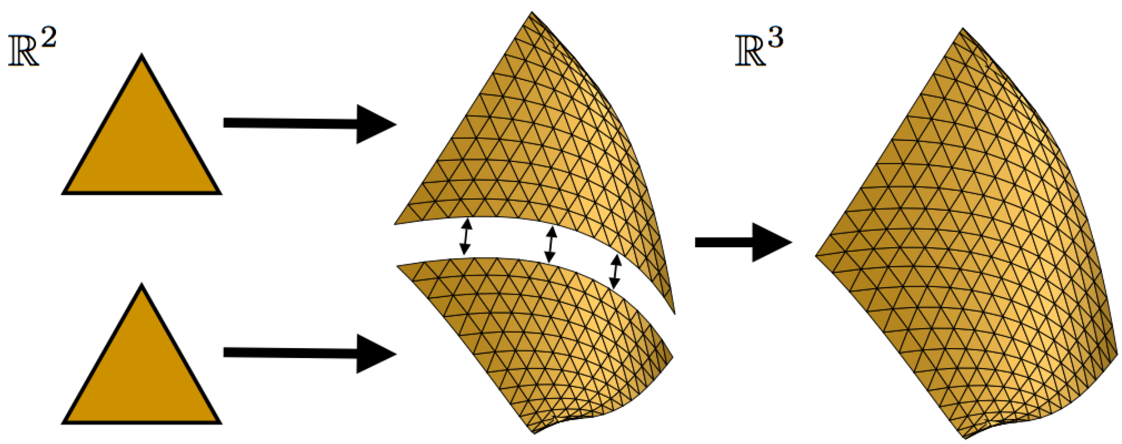

Figure 1.1: Two parametric surface patches joining together along their common boundary.

Subdivision surfaces are another common approach to fit a smooth surface to triangle or quadrilateral meshes of arbitrary topology, and they have been extensively investigated in the recent past [17, 18, 19, 20, 21, 22, 23]. These surfaces are limit surfaces obtained by repeatedly subdividing a given polygonal mesh. The subdivision process requires nothing else than vertex positions and connectivity information, is in general very simple, can easily handle meshes with arbitrary topology, and produces smooth surfaces with good visual quality in an intuitive sense, except near vertices of high degree[3]. These are the main reasons for the success of subdivision surfaces in the computer graphics and geometric modeling communities.

Despite their advantages for modeling surfaces of arbitrary topology, subdivision surfaces also have drawbacks. For instance, surface evaluation is often carried out by explicit, recursive subdivision, as most subdivision schemes do not possess a closed-form, analytic formulation (the Catmull-

[3]The degree of a vertex is the number of edges incident to it.

Clark subdivision scheme [17] is a notable exception [24]). In addition, most existing subdivision schemes yield G^k- or C^k-continuous surfaces, for $k = 1, 2$, only. If the input mesh has *extraordinary* vertices[4], then the resulting subdivision surface is not even C^1 at those vertices, and it may also present shape artifacts around them [25, 26]. Although it is possible to produce subdivision surfaces with C^2 or even higher continuity order at extraordinary vertices, previous efforts by Prautzsch and Reif [27, 28] have shown that subdivision schemes to produce such surfaces cannot be as simple and elegant as existing subdivision schemes. Finally, there is also no easy way to parametrize a subdivision surface for purposes such as texture mapping.

Implicit functions have also been used to fit smooth surfaces to triangle meshes [29, 30]. Implicit and parametric representations have complementary properties, and hence the advantages and drawbacks of each is highly dependent on the application [31]. In particular, implicit functions have been successfully used for fitting surfaces to dense and unorganized point sets [32, 33, 34]. This is because unorganized point sets have no explicit topological information, and this information is not required for defining an implicit surface that interpolates or closely approximates the points. However, in general the topology of the resulting surface cannot be anticipated, unless the point set is very dense and satisfies some special constraints [35]. Although the topology is known *a priori* in the surface fitting problem we are interested here (i.e., it is the mesh topology), ensuring that the implicit surface will have this exact topology remains very difficult, and we are not aware of any result that provides such a guarantee for triangle meshes of arbitrary topology.

Finally, a manifold-based approach pioneered by Grimm and Hughes [36] has proved to be well-suited to fit with relative ease, C^k-continuous parametric surfaces to triangle and quadrilateral meshes, for any arbitrary finite k or even $k = \infty$ [37, 38, 39]. Manifold-based constructions also share some of the most important properties of splines surfaces, such as local shape control and fixed-sized local support for basis functions. In addition, the differential structure of a manifold provides us with a natural setting for solving equations on surfaces with complex topology and geometry. Thus, as pointed out in [40], a manifold is a very attractive surface representation form for a handful of applications in computer graphics, such as reaction-diffusion texture [41], texture synthesis [42, 43], fluid simulation [44], and surface deformation [45].

We have designed a new manifold-based construction for fitting a C^∞-continuous surface to a triangle mesh of arbitrary topology. It turns out that a complete description and justification of this method is too lengthy to fit reasonably in a single article so this paper presents the theoretical framework that provides a sound justification for the correctness of our construction. This framework is a slight improvement upon the one in [36], which was also used to undergird the constructions in [37, 38]. A subsequent paper will present the details of our new construction and its implementation.

Our construction possesses most of the best features of each previous constructions. In particular, it is more compact and simpler than the one in [36], more powerful than the construction

[4]For triangle (resp. quadrilateral) mesh based schemes, this means a vertex of degree different from six (resp. four).

in [39], and shares with [38], a construction devised for quadrilateral meshes, the ability of producing C^∞-continuous surfaces and the flexibility in ways of defining the geometry of the resulting surface.

After a review of prior work, given in Chapter 2, we review some basic mathematical notions in Chapter 3, and introduce the theoretical framework (gluing data) that supports our manifold-based construction in Chapter 4. In Chapter 5 we describe in detail a new method for constructing sets of gluing data from a triangular mesh. Finally in Chapter 6, we offer some concluding remarks and directions for future work.

Chapter 2

Background and Prior Work

The formal definition of a manifold can be found in standard mathematics textbooks [46, 47, 48], and is also given in Chapter 3. Roughly speaking, manifolds are spaces that locally look like some n-dimensional Euclidean space, and on which we one can do calculus (e.g., compute derivatives, integrals, volumes, and curvatures). For that, each manifold, M, is equipped with a differentiable structure called an *atlas*. An atlas, $\mathcal{A}$, is a collection of *charts*. Each chart is a pair, (U, φ), where U is an open set of M and $\varphi : U \to \mathbb{R}^n$ is a continuous and bijective map whose inverse is also continuous. This means that $\varphi(U)$ is also an open set of $\mathbb{R}^n$. Furthermore, every point of the manifold, M, belongs to the open set, U, of at least one chart of its atlas, $\mathcal{A}$. Thus, the atlas, $\mathcal{A}$, establishes a correspondence between one neighborhood (i.e., some U) of every point of M and an open set (i.e., the set $\varphi(U)$) of $\mathbb{R}^n$. That's why we say that, locally, M looks like $\mathbb{R}^n$.

An atlas also enables us to do calculus on $\varphi(U)$ as we were doing on U. However, because the open sets, U_1 and U_2, of two distinct charts, (U_1, φ_1) and (U_2, φ_2), can overlap, we must also establish a correspondence between the subsets, $\varphi_1(U_1 \cap U_2)$ and $\varphi_2(U_1 \cap U_2)$, of $\mathbb{R}^n$ in order to do calculus on $\varphi_1(U_1)$ and $\varphi_2(U_2)$ in a consistent manner. This is done by defining *transition functions*, $\varphi_{21} : \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$ and $\varphi_{12} : \varphi_2(U_1 \cap U_2) \to \varphi_1(U_1 \cap U_2)$, which are required to satisfy the following two conditions (refer to Figure 2.1):

$$\varphi_{21} = \varphi_2 \circ \varphi_1^{-1} \quad \text{and} \quad \varphi_{12} = \varphi_1 \circ \varphi_2^{-1} .$$

Transition functions are usually required to be C^k-continuous, for some finite, non-negative integer k or even $k = \infty$, so that the necessary degree of "smoothness" to compute certain differential properties of M is ensured. Transition functions define which points in $\varphi_1(U_1 \cap U_2)$ and $\varphi_2(U_1 \cap U_2)$ are the "same", i.e., correspond to the same point in M under φ_1^{-1} and φ_2^{-1}. They also provide us with a means of "moving" along M without actually being on M, allowing us to consistently do global calculations on M.

Grimm and Hughes [36] offers a very elucidating real-world analogy to a manifold: portions of the earth, i.e., Europe (the open set U_1) and Asia (the open set U_2), are laid flat to paper maps (the open sets $\varphi_1(U_1)$ and $\varphi_2(U_2)$), as illustrated by Figure 2.2.

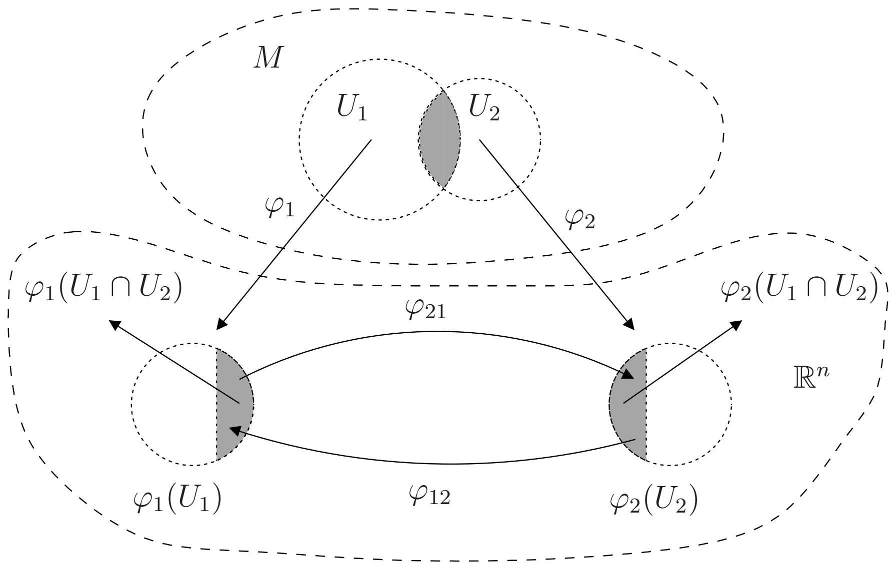

Figure 2.1: Illustration of the definition of a manifold.

Every bit of the world must be laid down to at least one paper map of the world atlas (i.e., every point of M belongs to an open set, U, of a chart). Overlaps of the open sets of two charts are represented by Europe and Asia both containing the country of Russia. The navigation from the map of Asia to the map of Europe does not require additional construct in real life, but is mathematically achieved via a transition function. Also, we can walk around the world, without being physically there, by moving from a position in one map to its counterpart position in an overlapping map.

A manifold-based approach for surface construction aims at building a manifold, M, which is a smooth surface in $\mathbb{R}^3$. For that, the definition of a manifold is not very helpful. The reason is that it departs from the fact that the manifold already exists. Fortunately, it is possible to define M in a constructive way from what we call a set of *gluing data* and a set of *parametrizations*. A set of gluing data consists of a collection of open sets in $\mathbb{R}^n$, called *parametrizations domains* (or *p-domains* for short), a collection of *gluing domains*, which are open subsets of p-domains, and a collection of transition functions, which are functions from gluing domains to gluing domains. In turn, each parametrization is a map from a p-domain to a subset, M, of $\mathbb{R}^m$. There is a simple correspondence between the constituents of the traditional definition of a manifold and the ones of a set of gluing data and a set of parametrizations (refer to Figure 2.3):

- each p-domain, $\Omega_i \subseteq \mathbb{R}^n$, is the image, $\Omega_i = \varphi_i(U_i)$, of an open set, U_i, of M under the map φ_i of the chart (U_i, φ_i) of an atlas of M;

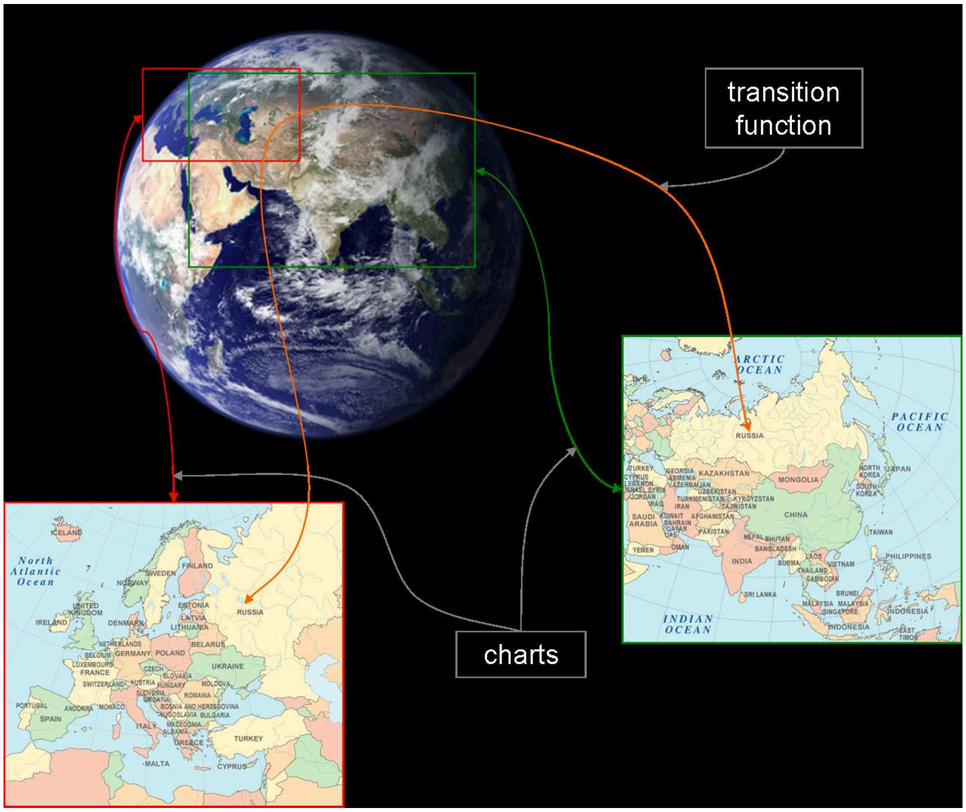

Figure 2.2: Manifold and the World Atlas

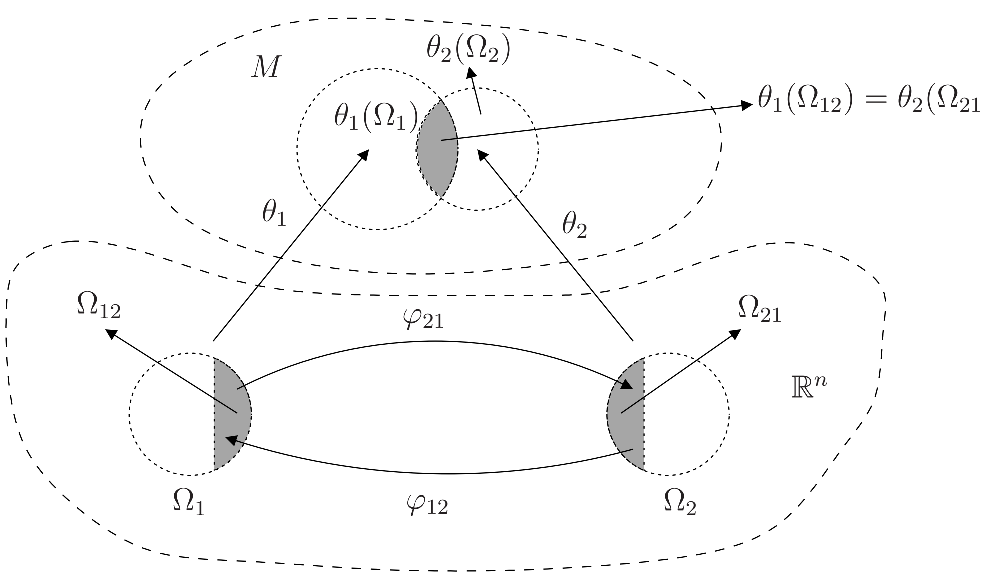

Figure 2.3: Illustration of p-domains, gluing domains, transition functions, and parametrizations.

- each gluing domain, $\Omega_{ij} \subseteq \Omega_i$, is the image, $\Omega_{ij} = \varphi_i(U_i \cap U_j)$, of the overlapping subset, $U_i \cap U_j$, of U_i and U_j;
- each transition function, $\varphi_{ji} : \Omega_{ij} \to \Omega_{ji}$, is a function from $\varphi_i(U_i \cap U_j) = \Omega_{ji}$ to $\varphi_j(U_i \cap U_j) = \Omega_{ij}$; and
- each parametrization, $\theta_i : \Omega_i \to M$, is the inverse, φ_i^{-1}, of the map $\varphi_i : U_i \to \mathbb{R}^n$, of the chart, (U_i, φ_i).

The key idea behind a manifold-based approach for surface construction is to define a set of gluing data and a set of parametrizations from the given triangle mesh. The idea of defining manifolds from a set of gluing data and a set of parametrizations is not new. André Weil introduced this idea to define abstract algebraic varieties by gluing irreducible affine sets in his book [49] published in 1946. The same idea is well-known in bundle theory and can be found in standard texts such as Steenrod [50], Bott and Tu [51], Morita [52], and Wells [53]. However, Grimm and Hughes [36, 54] were the first to have realized the power of the gluing process in surface modeling. We wish to emphasize that this is a very significant discovery and that their work inspired our construction, which is described in Chapter 5 of this paper.

The body of work on manifold-based constructions to surface modeling has been reviewed in detail in the recent SIGGRAPH 2006 course notes [40]. Grimm and Hughes [36] introduced the first manifold-based construction for surface modeling, and their basic framework has been adopted in almost all subsequent constructions [37, 38], including ours. In their basic framework, a set of gluing data is defined from the given mesh by associating p-domains with mesh vertices, edges, or triangles. Gluing domains and transition functions are determined by the mesh connectivity. Finally, a set of parametrizations is defined using the mesh geometry. The efficiency of a manifold-based construction depends upon the size of the set of gluing data and the complexity of the transition functions and parametrizations. The smaller the set of gluing data is and the simpler the transition functions and parametrizations are, the more efficient the construction is.

The construction in [36] takes a triangle mesh as input, subdivides the mesh by one step of the Catmull-Clark subdivision scheme, and then considers the dual of the subdivided mesh (which is no longer a triangle mesh). So, if the input mesh has v vertices, e edges, and t triangles, then the dual mesh will have $3v$ vertices, $3e$ edges, and $v + e + t$ faces. A set of gluing data is defined from the dual mesh by assigning a p-domain with each vertex, edge, and face of the mesh, which gives a total of $v + 4e + 4t$ p-domains. The p-domains associated with the vertices differ from the ones associated with the edges and faces, which in turn are also distinct. Furthermore, there are three distinct types of transition functions. The construction in [36] yields C^2-continuous surfaces only, but it was later simplified and improved [55] to produce C^k-continuous surfaces, for any finite integer k. Subsequent efforts [37, 38] aimed at providing a construction that requires a smaller set of gluing data, consists of simpler transition functions, and achieves C^∞-continuity.

Navau and Garcia [37] introduced a construction that takes a quadrilateral mesh and two integers, k and n, as input. The integer k specifies the finite degree of continuity of the resulting surface, while n is related to the extent of p-domains and gluing domains. The construction assigns a p-domain with each vertex of the mesh. A p-domain is said to be regular if its associated vertex is regular (i.e., the degree of the vertex is 4); otherwise, it is said to be irregular. Transition functions map gluing domains from regular to regular, regular (resp. irregular) to irregular (resp. regular), and irregular to irregular p-domains. So, like in [36], there are also three types of transition functions, but the one from regular to regular p-domains is trivial.

The size of the gluing data, however, depends on n and on the topology of the input mesh. This is because an irregular vertex cannot be in the neighborhood consisting of the $n + 1$ "layers" of quadrilaterals surrounding another irregular vertex. In addition, the graph consisting of the vertices and edges of the $n+1$ layers of quadrilaterals surrounding each vertex of the mesh must be planar. If any of these two requirements is not satisfied, the mesh is subdivided by the Catmull-Clark scheme, resulting in a larger mesh. So, for input triangle and quadrilateral meshes of comparable sizes, the construction in [37] may construct a set of gluing domain larger than the one constructed by the construction in [36]. This is true even for small values of n, with $n \geq 2$, as the quadrilateral mesh may contain an edge whose endpoints are irregular vertices.

Ying and Zorin [38] devised another manifold-based construction, which also takes a quadrilateral mesh as input and considerably improves upon the two previous constructions in several ways. First, the number of p-domains is fixed and equals the number of vertices of the input mesh (which is never subdivided). Second, there is only one type of transition function, which greatly simplifies their construction. Third, the resulting surface is C^∞-continuous. The construction in [38] offers a more flexible control of the geometry of the resulting surface than the ones in [36, 37].

More recently, Gu, He, and Qin [39] introduced another manifold-based construction for building smooth surfaces from triangle meshes. Unlike previous constructions [36, 37, 38], the construction in [39] is based on a novel theoretical framework, which undergirds what the authors called *manifold splines*. The main advantage of manifold splines over previous constructions is that their transition functions are affine and the parametrizations are either polynomial or rational polynomial functions. However, according to classical result from characteristic class theory [56], closed surfaces (except tori) cannot be covered by an *affine atlas*, i.e., an atlas in which every transition function is affine. In particular, such surfaces contain points, called *singular points* or *singularities*, that cannot belong to the open set of any chart of any affine atlas.

The construction in [39] yields manifold splines with at most $2g-2$ singular points, where g is the genus of the input triangle mesh. The resulting manifold splines have two main drawbacks. First, they are difficult to construct in the neighborhood of singular points, and they are not differentiable there. Second, there are distortions in the parametrizations near singular points, which significantly affect the visual quality of the surface. Furthermore, the algorithm for constructing manifold splines is based on the computation of holomorphic 1-forms, which is equivalent to solving an elliptic partial differential equation on the mesh using the finite element method [57]. So, even though the transition functions used by the construction in [39] are simpler than the ones in [36, 37, 38], its set of gluing data is more complicated to compute.

An improvement upon the construction in [39] was recently described in [58]. By using the concept of discrete Ricci flow, the improved construction computes a metric on a parametric domain for the mesh. The parametric domain is computed by a global parametrization procedure that requires the mesh be cut open along a set of closed curves [57]. This metric induces an affine atlas covering the entire manifold, except for one singular point. A single point is the theoretical lower bound for the number of singular points. So, the construction in [58] is optimal as far as affine atlases are concerned. However, the complexity of the construction of its set of gluing data, which involves mesh segmentation and parametrization, remains large when compared to the complexity of the constructions in [36, 37, 38]. Moreover, the problems caused by singular points on the manifold splines are reduced to one neighborhood of the surface, but they are not eliminated.

The new manifold-based construction described here is also based on the basic framework adopted by the constructions in [36, 37, 38]. Our construction shares with the one in [38] its main improvements upon the constructions in [36, 37], namely: (1) it is simpler than the constructions in [36, 37], as there is only type of p-domain and only one type of transition function, and the number of p-domains (resp. parametrizations) is fixed and equals the number of vertices; (2) the resulting surface is C^∞-continuous.

One of the main differences from our construction to the one in [38] is that ours was devised to work with triangle meshes, which are far more popular than quadrilateral meshes in computer graphics and geometry processing applications [59].

Chapter 3

Mathematical Preliminaries

This chapter introduces basic mathematical concepts that are important for the understanding of our manifold-based construction. Most concepts were borrowed from standard textbooks on differentiable manifolds, such as [46, 47, 48].

3.1 Simplicial Surfaces

The input of the problem we are dealing with in this manuscript, a triangle mesh, is formally known as a *simplicial surface*. The goal of this section is to introduce the formal definition of a simplicial surface as well as some of its important properties. All concepts presented in this section can be found in the book by Bloch [60].

Definition 3.1. Let $v_0, \ldots, v_d$ be any $d+1$ affinely independent points in $\mathbb{R}^n$, where d is a non-negative integer. The *simplex* σ spanned by the points $v_0, \ldots, v_d$ is the convex hull of these points, and is denoted by $[v_0, \ldots, v_d]$. The points $v_0, \ldots, v_d$ are called the *vertices* of σ. The *dimension* of σ, denoted by $dim(\sigma)$, is d, and σ is called a d-simplex.

In $\mathbb{R}^n$, the largest number of affinely independent points is $n+1$, and we have simplices of dimension $0, 1, \ldots, n$. Note that a 0-simplex is a single point, a 1-simplex is a line segment, a 2-simplex is a triangle, and a 3-simplex is a tetrahedron. Note also that the convex hull of any non-empty subset of vertices of a simplex is again a simplex. This is a generalization of the observation that the boundary of a triangle consists of edges and vertices, and these edges and vertices are spanned by subsets of the vertices of the triangle.

Definition 3.2. Let $\sigma = [v_0, \ldots, v_d]$ be a d-simplex in $\mathbb{R}^n$. A *face* of σ is a simplex spanned by a non-empty subset of $\{v_0, \ldots, v_d\}$; if this subset is proper the face is called a *proper face*. A face of σ that is a k-simplex, where k is a non-negative integer, is called a *k-face*. The *combinatorial* *boundary of* σ, denoted by $bd(\sigma)$, is the union of all proper faces of σ. The *combinatorial interior of* σ, denoted by $int(\sigma)$, is defined to be $\sigma - bd(\sigma)$.

Simplices are used as building blocks for defining simplicial complexes, which are the most general objects we can construct from simplices. Simplicial complexes are built by gluing simplices together along their common faces. A simplicial surface is a particular type of simplicial complex built out of vertices, edges, and triangles. In what follows we give a definition of simplicial complex and some related concepts:

Definition 3.3. A *simplicial complex* $\mathcal{K}$ in $\mathbb{R}^n$ is a finite collection of simplices in $\mathbb{R}^n$ such that

(i) if a simplex is in $\mathcal{K}$, then all its faces are in $\mathcal{K}$;

(ii) if $\sigma, \tau \in \mathcal{K}$ are simplices such that $\sigma \cap \tau \neq \emptyset$, then $\sigma \cap \tau$ is a face of each σ and τ.

The *dimension* of $\mathcal{K}$, denoted by $dim(\mathcal{K})$, is the largest dimension of a simplex in $\mathcal{K}$, i.e., $dim(\mathcal{K}) = \max\{dim(\sigma) \mid \sigma \in \mathcal{K}\}$. We refer to a d-dimensional simplicial complex as simply a *d-complex*. The set consisting of the union of all points in the simplices of $\mathcal{K}$ is called the *underlying space* of $\mathcal{K}$, and it is denoted by $|\mathcal{K}|$.

Figure 3.1 shows three sets of simplices in $\mathbb{R}^2$. The set on the left is not a simplicial complex because it is missing an edge and a vertex. The set in the middle contains two simplices that intersect each other but the intersection is not a face of either one, and therefore it cannot be a simplicial complex. The set on the right is a simplicial complex. Note that the underlying space, $|\mathcal{K}|$, of any simplicial complex, $\mathcal{K}$, is a compact set, for K is a finite collection of simplices.

Definition 3.4. Let $\mathcal{K}$ be a simplicial complex in $\mathbb{R}^n$. For each integer i, with $0 \leq i \leq dim(\mathcal{K})$, we define $\mathcal{K}^{((i))}$ to be the collection of all i-simplices of $\mathcal{K}$.

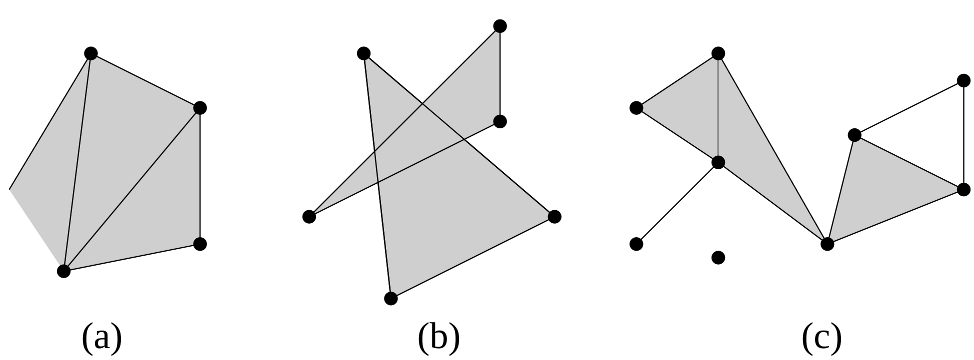

Figure 3.1: Collections of simplices in $\mathbb{R}^2$. (a) and (b) are not simplicial complexes, but (c) is.

Definition 3.5. Let $\mathcal{K}$ be a simplicial complex in $\mathbb{R}^n$. Then, if σ is a simplex in $\mathcal{K}$, the *star* and *link* of σ, denoted $st(\sigma, \mathcal{K})$ and $lk(\sigma, \mathcal{K})$, respectively, are defined to be

$$st(\sigma, \mathcal{K}) = \{\tau \in \mathcal{K} \mid \exists \eta \text{ in } \mathcal{K} \text{ such that } \sigma \text{ is a face of } \eta \text{ and } \tau \text{ is a face of } \eta\}$$

and

$$lk(\sigma, \mathcal{K}) = \{\tau \in \mathcal{K} \mid \tau \text{ is in } st(\sigma, \mathcal{K}) \text{ and } \tau \text{ and } \sigma \text{ have no face in common}\}.$$

Let $\mathcal{K}$ be the simplicial complex in Figure 3.2(a). Then, $\mathcal{K}^{((0))}$ consists of the 0-simplices $[p]$, $[q]$, $[r]$, $[s]$, $[t]$, $[u]$, $[v]$, $[x]$, $[y]$, and $[z]$; $\mathcal{K}^{((1))}$ consists of the 1-simplices $[p, q]$, $[p, s]$, $[p, v]$, $[q, r]$, $[q, s]$, $[r, s]$, $[r, v]$, $[s, v]$, $[t, u]$, $[t, v]$, $[t, x]$, $[u, x]$, $[v, x]$, $[v, z]$, $[x, y]$, $[x, z]$, and $[y, z]$; and $\mathcal{K}^{((2))}$ consists of the 2-simplices $[p, q, s]$, $[p, s, v]$, $[q, r, s]$, $[r, s, v]$, $[t, u, x]$, $[t, x, v]$, $[x, z, v]$, and $[x, y, z]$. The star $st([v], \mathcal{K})$ of $[v]$ consists of $[v]$, $[r]$, $[s]$, $[p]$, $[z]$, $[z]$, $[x]$, and $[t]$; 1-simplices $[p, v]$, $[r, v]$, $[r, s]$, $[s, p]$, $[s, v]$, $[t, v]$, $[x, v]$, $[z, v]$, $[z, v]$, $[z, x]$, and $[x, t]$; and 2-simplices $[r, s, v]$, $[p, s, v]$, $[t, v, x]$, and $[x, z, v]$, as illustrated by Figure 3.2(b). The link $lk([v], \mathcal{K})$ of $[v]$ consists of the 0-simplices $[p]$, $[r]$, $[s]$, $[t]$, $[x]$, $[z]$, and 1-simplices $[p, s]$, $[r, s]$, $[x, z]$, and $[t, x]$, as illustrated by Figure 3.2(c).

Definition 3.6. A 2-complex $\mathcal{K}$ is called a *simplicial surface* if every 1-simplex of $\mathcal{K}$ is the face of precisely two simplices of $\mathcal{K}$, and the underlying space of the link of each 0-simplex of $\mathcal{K}$ is homeomorphic to the unit 1-sphere, $\mathbb{S}^1 = \{x \in \mathbb{R}^2 \mid \|x\| = 1\}$. The underlying space of a simplicial surface is called the *underlying surface* of the simplicial surface.

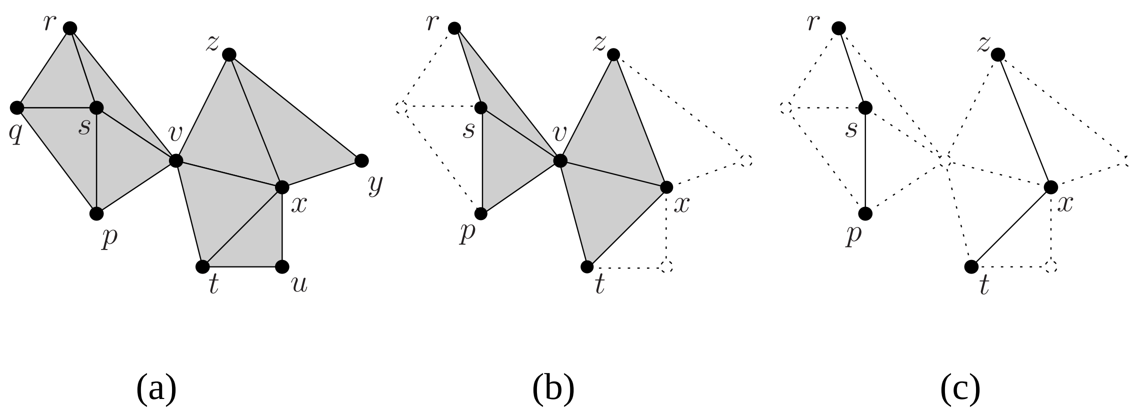

Figure 3.2: (a) A simplicial complex. (b) The star of vertex v in (a). (c) The link of vertex v in (a).

For instance, the simplicial complex consisting of all proper faces of a tetrahedron is a simplicial surface. However, the simplicial complex consisting of all proper faces of the two tetrahedra in Figure 3.3 is not a simplicial surface, as the link of $[v]$ is not homeomorphic to $\mathbb{S}^1$. Recall that a subset $S \subset \mathbb{R}^n$ is called a topological surface (or surface, for short) if for every point $p \in S$, there exists an open ball, $B_\delta(p, \mathbb{R}^n)$, in $\mathbb{R}^n$, centered at p and with radius δ, where $\delta \in \mathbb{R}$ and $\delta > 0$, such that $B_\delta(p, \mathbb{R}^n) \cap S$ is homeomorphic to the open unit disk, $\mathbb{D} = \{p \in \mathbb{R}^2 \mid \|p\| < 1\}$, in $\mathbb{R}^2$. The following lemma from [60] states an important property of simplicial surfaces:

Lemma 3.1. *Let $\mathcal{K}$ be a simplicial complex in $\mathbb{R}^n$. Then $|\mathcal{K}|$ is a topological surface if and only if $\mathcal{K}$ is a simplicial surface.*

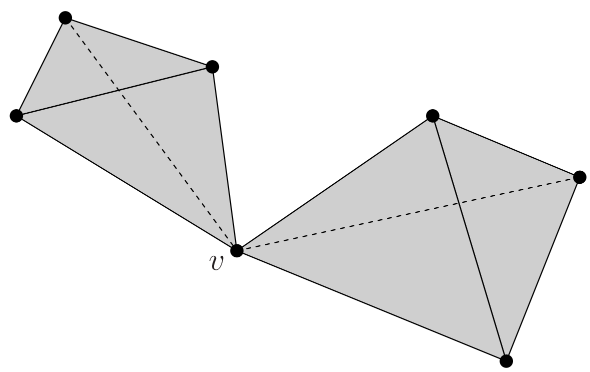

Figure 3.3: The 2-complex consisting of the proper faces of the two tetrahedra is not a simplicial surface.

Definition 3.7. Let $\mathcal{K}$ be a simplicial complex in $\mathbb{R}^n$, and let $\mathcal{L}$ be a simplicial complex in $\mathbb{R}^m$. A map $f : \mathcal{K}^{((0))} \to \mathcal{L}^{((0))}$ is a *simplicial map* if whenever $[v_0, \ldots, v_d]$ is a simplex in $\mathcal{K}$, then $[f(v_0), \ldots, f(v_d)]$ is a simplex in $\mathcal{L}$. A simplicial map is a *simplicial isomorphism* if it is a bijective map on the set of vertices, and if its inverse is also a simplicial map. If there is a simplicial isomorphism from $\mathcal{K}$ to $\mathcal{L}$ then we say that $\mathcal{K}$ and $\mathcal{L}$ are *simplicially isomorphic*.

For instance, let $\mathcal{K}$ be a tetrahedron. Since any subset of two or three vertices of $\mathcal{K}$ is the set of vertices of a simplex in $\mathcal{K}$, it follows that any map $f : \mathcal{K}^{((0))} \to \mathcal{K}^{((0))}$ is a simplicial map, which is also a simplicial isomorphism.

3.2 Topological Spaces and Homeomorphisms

Definition 3.8. Let M be a set. A *topology* on M is a collection $\mathcal{T}_M$ of subsets of M satisfying three axioms:

(1) $\emptyset$ and M belong to $\mathcal{T}_M$;

(2) if $U_1, \ldots, U_n \in \mathcal{T}_M$ then $\left(\bigcap_{i=1}^n U_i\right) \in \mathcal{T}_M$; and

(3) if I is any (possibly infinite) indexing set and $U_i \in \mathcal{T}_M$, for all $i \in I$, then $\left(\bigcup_{i \in I} U_i\right) \in \mathcal{T}_M$.

Each $U \in \mathcal{T}_M$ is called an *open set* of $\mathcal{T}_M$. In short, a topology on M is a family of subsets of M (the *open sets*), containing $\emptyset$ and M, which is closed under the operation of union and finite intersection. A *topological space* is a pair, $(M, \mathcal{T}_M)$, consisting of a set, M, and a topology, $\mathcal{T}_M$, on M. We often speak of the topological space M and its open sets, omitting $\mathcal{T}_M$ from the notation when it is clear what topology is intended.

For instance, the set $\mathbb{R}^n$ is often regarded as a topological space equipped with the "usual" topology: the open sets are $\mathbb{R}^n$, $\emptyset$, and all nonempty proper subsets $U \subset \mathbb{R}^n$ such that for every $p = (p_1, \ldots, p_n) \in U$, there exists a real number δ, with $\delta > 0$, such that the *open ball,* $B_\delta(p, \mathbb{R}^n)$, *in* $\mathbb{R}^n$ *of center* p *and radius* δ, i.e.,

$$B_\delta(p, \mathbb{R}^n) = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid \left(\sum_{i=1}^{n} (x_i - p_i)^2\right) < \delta^2\},$$

is a subset of U. It can be shown that the "usual" topology is indeed a topology, i.e., it satisfies conditions (1)-(3) of Definition 3.8.

Definition 3.9. If M and N are topological spaces[1], a function $f : M \to N$ is *continuous* if, for every open set $U \subset N$, the set $f^{-1}(U) \subset M$ is also open. A function $f : M \to N$ is a *homeomorphism* if f is bijective, and both f and f^{-1} are continuous. If $f : M \to N$ is a homeomorphism, we say that M and N are *homeomorphic*, and we denote this fact by $M \simeq N$.

3.3 Manifolds

Given $\mathbb{R}^n$, recall that the projection functions, $pr_i : \mathbb{R}^n \to \mathbb{R}$, are defined by

$$pr_i(x_1, \ldots, x_n) = x_i, \quad \text{for all } 1 \leq i \leq n.$$

Definition 3.10. Given a topological space, M, a *chart* (or *local coordinate function*) is a pair, (U, φ), where U is an open subset of M and $\varphi : U \to \Omega$ is a homeomorphism onto an open subset, $\Omega = \varphi(U)$, of $\mathbb{R}^{n_\varphi}$ (for some $n_\varphi \geq 1$). For any $p \in M$, a chart, (U, φ), is a *chart at* p if and only if $p \in U$. If (U, φ) is a chart, then the functions $x_i = pr_i \circ \varphi$ are called *local coordinates* and for every $p \in U$, the tuple $(x_1(p), \ldots, x_n(p))$ is the set of *coordinates of* p with respect to the chart. The pair (Ω, φ^{-1}), the "inverse" of (U, φ), is called a *local parametrization.*

Definition 3.11. Given a topological space, M, and any two charts, (U_1, φ_1) and (U_2, φ_2), where U_1 and U_2 are open subsets of M, if $U_1 \cap U_2 \neq \emptyset$, we define the *transition functions*, $\varphi_{ji} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ and $\varphi_{ij} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$, as

$$\varphi_{ji} = \varphi_j \circ \varphi_i^{-1} \quad \text{and} \quad \varphi_{ij} = \varphi_i \circ \varphi_j^{-1}.$$

[1] Notice that we are already omitting mention of the topologies $\mathcal{T}_M$ and $\mathcal{T}_N$.

Figure 2.1 illustrates Definition 3.11.

Note that $\varphi_{ij} = (\varphi_{ji})^{-1}$ and that the transition functions φ_{ji} (resp. φ_{ij}) are functions between open sets of $\mathbb{R}^n$. This is good news, as the whole arsenal of calculus is available for functions on $\mathbb{R}^n$, and many important results of calculus can be promoted to manifolds by imposing suitable conditions on transition functions.

Definition 3.12. Given a topological space, M, given some integer $n \geq 1$, and given some k such that k is either an integer, with $k \geq 1$, or $k = \infty$, a C^k *n-atlas* (or *n-atlas of class* C^k), $\mathcal{A}$, on M is a family of charts, $\{(U_i, \varphi_i)\}_{i\in I}$, where I is a non-empty (possibly infinite) countable set, such that the following holds:

(1) $\varphi_i(U_i) \subseteq \mathbb{R}^n$, for all i;

(2) the family $\{U_i\}_{i\in I}$ is an open cover for M, i.e.,

$$M = \bigcup_{i\in I} U_i\,;$$

and

(3) whenever $U_i \cap U_j \neq \emptyset$, the transition function φ_{ji} (resp. φ_{ij}) is a C^k diffeomorphism.

For an example, consider the sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$,

$$\mathbb{S}^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} x_i^2 = 1\}.$$

We can regard $\mathbb{S}^n$ as a topological space by giving $\mathbb{S}^n$ the topology consisting of all subsets U of $\mathbb{S}^n$ such that, for every $p = (p_1, \ldots, p_{n+1}) \in U$, there exists a real number δ, with $\delta > 0$, such that $(\mathbb{S}^n \cap B_\delta(p, \mathbb{R}^{n+1})) \subseteq U$, where $B_\delta(p, \mathbb{R}^{n+1})$ is the open ball in $\mathbb{R}^{n+1}$ of center p and radius δ. Using the stereographic projections (from the north pole and south pole), we can define two charts on $\mathbb{S}^n$. Denote the points $(0, \ldots, 0, 1) \in \mathbb{R}^{n+1}$ and $(0, \ldots, 0, -1) \in \mathbb{R}^{n+1}$ by N (the north pole) and S (the south pole), respectively, and let $\varphi_N : \mathbb{S}^n - \{N\} \to \mathbb{R}^n$ and $\varphi_S : \mathbb{S}^n - \{S\} \to \mathbb{R}^n$ be the functions

$$\varphi_N(x_1, \ldots, x_{n+1}) = \frac{1}{1 - x_{n+1}}(x_1, \ldots, x_n) \quad \text{and} \quad \varphi_S(x_1, \ldots, x_{n+1}) = \frac{1}{1 + x_{n+1}}(x_1, \ldots, x_n)\,,$$

which are called *stereographic projection from the north pole* and *stereographic projection from the south pole*, respectively. The inverse stereographic projections are given by

$$\varphi_N^{-1}(x_1, \ldots, x_n) = \frac{1}{\left(\sum_{i=1}^n x_i^2\right) + 1}\left(2x_1, \ldots, 2x_n, \left(\sum_{i=1}^n x_i^2\right) - 1\right)$$

and

$$\varphi_S^{-1}(x_1, \ldots, x_n) = \frac{1}{\left(\sum_{i=1}^n x_i^2\right) + 1}\left(2x_1, \ldots, 2x_n, -\Big(\sum_{i=1}^n x_i^2\Big) + 1\right).$$

Note that φ_N and φ_S are homeomorphisms that map open sets of $\mathbb{S}^n$ to open sets of $\mathbb{R}^n$ (regarding $\mathbb{R}^n$ as a topological space equipped with the usual topology). So, (U_N, φ_N) and (U_S, φ_S) are charts. Furthermore, if we let $U_N = \mathbb{S}^n - \{N\}$ and $U_S = \mathbb{S}^n - \{S\}$, we see that (1) $\varphi_N(U_N) = \mathbb{R}^n$ and $\varphi_S(U_S) = \mathbb{R}^n$, (2) $\{U_N, U_S\}$ is an open cover for $\mathbb{S}^n$, and (3) it is easily checked that on the overlap, $U_N \cap U_S = \mathbb{S}^n - \{N, S\}$, the transition functions,

$$\varphi_{SN} = \varphi_S \circ \varphi_N^{-1} \quad \text{and} \quad \varphi_{NS} = \varphi_N \circ \varphi_S^{-1}$$

are given by

$$(x_1, \ldots, x_n) \mapsto \frac{1}{\sum_{i=1}^n x_i^2}(x_1, \ldots, x_n),$$

which is a smooth bijection on $\mathbb{R}^n - \{O\}$. So, we conclude that (U_N, φ_N) and (U_S, φ_S) form a smooth n-atlas on $\mathbb{S}^n$.

The existence of a C^k n-atlas on a topological space, M, is sufficient to establish that M is an n-dimensional C^k manifold, but there is still a minor subtlety in the actual definition of a manifold. This has to do with the fact that there may be many choices of atlases, but it is useful to think of a manifold as an object independent of the choice of atlas. To do so, we define the notion of atlas compatibility. Given a C^k n-atlas, $\mathcal{A}$, on M, for any other chart, (U, φ), we say that (U, φ) is *compatible* with the atlas $\mathcal{A}$ if and only if every function $\varphi_i \circ \varphi^{-1}$ (resp. $\varphi \circ \varphi_i^{-1}$) is C^k (whenever $U \cap U_i \neq \emptyset$). Two atlases, $\mathcal{A}$ and $\mathcal{A}'$, on M are *compatible* if and only if every chart of one atlas is compatible with the other atlas. This is equivalent to saying that the union of the two atlases is still an atlas. It can be shown that compatibility induces an equivalence relation on C^k n-atlases on M. In fact, given an atlas, $\mathcal{A}$, on M, the collection, $\tilde{\mathcal{A}}$, of all charts compatible with $\mathcal{A}$ is a maximal atlas in the equivalence class of charts compatible with $\mathcal{A}$. Finally, we define a manifold as follows:

Definition 3.13. Given an integer $n \geq 1$ and given some k such that k is either an integer, with $k \geq 1$, or $k = \infty$, a C^k *manifold of dimension* n consists of a topological space, M, together with an equivalence class, $\overline{\mathcal{A}}$, of C^k n-atlases on M. Any atlas, $\mathcal{A}$, of $\overline{\mathcal{A}}$ is called a *differentiable structure of class* C^k (*and dimension* n) on M. When $k = \infty$, we say that M is a *smooth manifold.*

To avoid pathological cases and to ensure that a manifold is embeddable in $\mathbb{R}^n$, for some $n \geq 1$, we require that the topology of M be *Hausdorff* and *second-countable*. Hausdorff means that for every distinct points, $x \neq y$ in M, there are disjoint open subsets, U_x and U_y, with $x \in U_x$ and $y \in U_y$. Second-countable means that there is a countable set of open subsets of M such that every open subset of M is a union of opens from this countable set. Thus, as it is customary, in this paper, manifolds are required to be Hausdorff and second-countable.

Definition 3.13 relates to our informal discussion in Chapter 2 as follows: The manifold, M, can be viewed as the world; an atlas $\mathcal{A}$ on M correspond to a collection of regions of the world (the open sets $\{U_i\}_{i\in I}$), so that each region U_i is associated with a map, $\varphi_i : U_i \to \Omega_i$, from the region to a rectangular page of the World Atlas, Ω_i; and the functions φ_{ji} and φ_{ij} provide us with a way of moving from one page to another page of the World Atlas in a consistent manner. In particular, given the "local coordinates" of a location, p, in a rectangular page, $\Omega_i = \varphi(U_i)$, of the world atlas, we can move to another page of the atlas, say $\Omega_j = \varphi(U_j)$, which covers another region, U_j, of the world containing $\varphi_i^{-1}(p)$ (i.e., $\varphi_i^{-1}(p) \in (U_i \cap U_j)$), by using φ_{ji}. The transition φ_{ji} can be viewed as a two-step move: (1) go from the World Atlas to the world using φ_i^{-1} and then (2) return to the atlas page, $\Omega_j = \varphi_j(U_j)$, that covers U_j using φ_j. However, once we have φ_{ji}, we do not need the world in order to moving from one page to another page of the World Atlas. This is actually the key idea behind the *gluing process* for constructing manifolds from sets of gluing data.

Chapter 4

Construction of Manifolds from Gluing Data

4.1 Sets of Gluing Data for Manifolds

Recall that the goal of this work is to build a C^k surface, S, where $S \subset \mathbb{R}^3$, and $k \geq 1$ or $k = \infty$, that approximates the underlying surface of a given simplicial surface in $\mathbb{R}^3$. To that end, we propose a new construction that defines the surface S as a manifold. However, for our purposes, the traditional definition of a manifold (see Definition 3.13) is not very helpful. The reason is that the standard definition assumes that the object we want to build, the manifold, already exists. Remarkably, manifolds can also be defined by a gluing process, using what is often called a set of gluing data. In what follows, we define the notion of gluing data and show that it is possible, in principle, to construct a manifold from any given set of gluing data.

One of the main difficulties is to ensure that the space obtained by gluing the pieces Ω_{ij} and Ω_{ji} is Hausdorff. Some care must also be exercised in formulating the consistency conditions relating the φ_{ji}'s (the so-called "cocycle condition"). This is because the traditional condition used in bundle theory (for example, see Steenrod [50] or Bott and Tu [51]) has to do with triple overlaps of the $U_i = \varphi_i^{-1}(\Omega_i)$ on the manifold, M, but in our situation, we do not have M nor the parametrization maps $\theta_i = \varphi_i^{-1}$ and the cocycle condition on the φ_{ji}'s has to be stated in terms of the Ω_i's and the Ω_{ji}'s.

Finding an easily testable necessary and sufficient criterion for the Hausdorff condition is a subtle problem. We propose a necessary and sufficient condition but it is not easily testable in general, although it is easy to check for the construction given in Chapter 5.

If M is a manifold, then observe that difficulties may arise when we want to separate two distinct point, $p, q \in M$, such that p and q neither belong to the same open, $\theta_i(\Omega_i)$, nor to two disjoint opens, $\theta_i(\Omega_i)$ and $\theta_j(\Omega_j)$, but instead, to the boundary points in $(\partial(\theta_i(\Omega_{ij}))\cap\theta_i(\Omega_i))\cup(\partial(\theta_j(\Omega_{ji}))\cap\theta_j(\Omega_j))$. In this case, there are some disjoint open subsets, U_p and U_q, of M with $p \in U_p$ and $q \in U_q$, and we get two disjoint open subsets, $V_x = \theta_i^{-1}(U_p) \subseteq \Omega_i$ and $V_y = \theta_j^{-1}(U_q) \subseteq \Omega_j$, with $\theta_i(x) = p$, $\theta_j(y) = q$, and such that $x \in \partial(\Omega_{ij}) \cap \Omega_i$, $y \in \partial(\Omega_{ji}) \cap \Omega_j$, and no point in $V_y \cap \Omega_{ji}$ is the image of any point in $V_x \cap \Omega_{ij}$ by φ_{ji}. Since V_x and V_y are open, we may assume that they are open balls. This necessary condition turns out to be also sufficient.

With the above motivations in mind, here is the definition of sets of gluing data.

Definition 4.1. Let n be an integer with $n \geq 1$ and let k be either an integer with $k \geq 1$ or $k = \infty$. A *set of gluing data* is a triple,

$$\mathcal{G} = \left((\Omega_i)_{i\in I}, (\Omega_{ij})_{(i,j)\in I\times I}, (\varphi_{ji})_{(i,j)\in K}\right),$$

satisfying the following properties, where I and K are (possibly infinite) countable sets, and I is non-empty:

(1) For every $i \in I$, the set Ω_i is a non-empty open subset of $\mathbb{R}^n$ called *parametrization domain*, for short, *p-domain*, and the Ω_i are pairwise disjoint (i.e., $\Omega_i \cap \Omega_j = \emptyset$ for all $i \neq j$).

(2) For every pair $(i, j) \in I \times I$, the set Ω_{ij} is an open subset of Ω_i. Furthermore, $\Omega_{ii} = \Omega_i$ and $\Omega_{ji} \neq \emptyset$ if and only if $\Omega_{ij} \neq \emptyset$. Each non-empty Ω_{ij} (with $i \neq j$) is called a *gluing domain*.

(3) If we let
$$K = \{(i, j) \in I \times I \mid \Omega_{ij} \neq \emptyset\},$$
then $\varphi_{ji} : \Omega_{ij} \to \Omega_{ji}$ is a C^k bijection for every $(i, j) \in K$ called a *transition function* (or *gluing function*) and the following conditions hold:

    (a) $\varphi_{ii} = \mathrm{id}_{\Omega_i}$, for all $i \in I$,

    (b) $\varphi_{ij} = \varphi_{ji}^{-1}$, for all $(i, j) \in K$, and

    (c) For all i, j, k, if $\Omega_{ji} \cap \Omega_{jk} \neq \emptyset$, then $\varphi_{ji}^{-1}(\Omega_{ji} \cap \Omega_{jk}) \subseteq \Omega_{ik}$ and $\varphi_{ki}(x) = \varphi_{kj} \circ \varphi_{ji}(x)$, for all $x \in \varphi_{ji}^{-1}(\Omega_{ji} \cap \Omega_{jk})$ (see Figure 4.1).

(4) For every pair $(i, j) \in K$, with $i \neq j$, for every $x \in \partial(\Omega_{ij}) \cap \Omega_i$ and $y \in \partial(\Omega_{ji}) \cap \Omega_j$, there are open balls, V_x and V_y, centered at x and y, so that no point of $V_y \cap \Omega_{ji}$ is the image of any point of $V_x \cap \Omega_{ij}$ by φ_{ji} (see Figure 4.2).

We can think of the p-domains Ω_i as the images $\varphi_i(U_i)$ of the charts (U_i, φ_i) of the manifold, M, we want to define. Likewise, we can think of the gluing domains Ω_{ij} and Ω_{ji} as the images $\varphi_i(U_i \cap U_j)$ and $\varphi_j(U_i \cap U_j)$, under the maps φ_i and φ_j, of the overlap region $U_i \cap U_j$, respectively. Finally, the gluing functions $\varphi_{ji} : \Omega_{ij} \to \Omega_{ji}$ can be thought of as the transition functions of M.

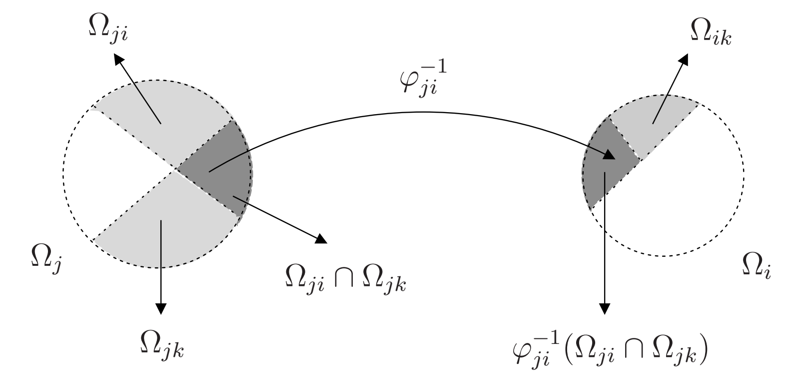

Figure 4.1: Illustration of condition 3(c) of Definition 4.1.

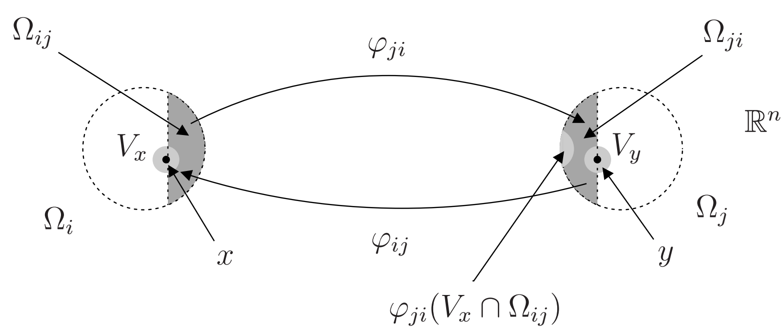

Figure 4.2: Illustration of condition 4 of Definition 4.1.

Observe that $\Omega_{ij} \subseteq \Omega_i$ and $\Omega_{ji} \subseteq \Omega_j$. If $i \neq j$, as Ω_i and Ω_j are disjoint, so are Ω_{ij} and Ω_{ji}. Observe also that both conditions 3(a) and 3(b) of Definition 4.1 follow from 3(c). More specifically, to get 3(a), set $i = j = k$ in 3(c). Then, 3(b) follows from 3(a) and 3(c) by setting $k = i$. Condition 3(c) is called the *cocycle condition* and it plays a crucial role in Theorem 4.1, which states that an n-dimensional C^k manifold can be constructed from the set of gluing data, $\mathcal{G}$. This condition may seem overly complicated, but it is actually needed to guarantee the transitivity of the relation, $\sim$, defined in the proof of Theorem 4.1. The problem is that $\varphi_{kj} \circ \varphi_{ji}$ is a partial function whose domain, $\varphi_{ji}^{-1}(\Omega_{ji} \cap \Omega_{jk})$, is not necessarily related to the domain, Ω_{ik}, of φ_{ki}. Consequently, in order to ensure the transitivity of $\sim$, we must assert that whenever the composition $\varphi_{kj} \circ \varphi_{ji}$ has nonempty domain, this domain is contained in the domain of φ_{ki} and that $\varphi_{kj} \circ \varphi_{ji}$ and φ_{ki} agree.

Theorem 4.1. For every set of gluing data,

$$\mathcal{G} = \big((\Omega_i)_{i\in I}, (\Omega_{ij})_{(i,j)\in I\times I}, (\varphi_{ji})_{(i,j)\in K}\big)\,,$$

there is an n-dimensional C^k manifold, $M_{\mathcal{G}}$, whose transition functions are the φ_{ji}'s.

Proof. Define the binary relation, $\sim$, on the disjoint union, $\coprod_{i\in I} \Omega_i$, of the open sets, Ω_i, as follows: For all $x, y \in \coprod_{i\in I} \Omega_i$,

$$x \sim y \quad \text{iff} \quad (\exists (i,j) \in K)(x \in \Omega_{ij}, y \in \Omega_{ji}, y = \varphi_{ji}(x)).$$

Note that if $x \sim y$ and $x \neq y$, then $i \neq j$, as $\varphi_{ii} = \mathrm{id}$. But then, as $x \in \Omega_{ij} \subseteq \Omega_i$, $x \in \Omega_{ji} \subseteq \Omega_j$ and $\Omega_i \cap \Omega_j = \emptyset$ when $i \neq j$, if $x \sim y$ and $x, y \in \Omega_i$, then $x = y$. We claim that $\sim$ is an equivalence relation. This follows easily from the co-cocycle condition but to be on the safe side, we provide the crucial step of the proof. Clearly, condition 3(a) of Definition 4.1 ensures reflexivity and condition 3(b) ensures symmetry. The crucial step is to check transitivity. Assume that $x \sim y$ and $y \sim z$. Then, there are some i, j, k such that

(i) $x \in \Omega_{ij}$, $y \in \Omega_{ji} \cap \Omega_{jk}$, $z \in \Omega_{kj}$, and

(ii) $y = \varphi_{ji}(x)$ and $z = \varphi_{kj}(y)$.

Consequently, $\Omega_{ji} \cap \Omega_{jk} \neq \emptyset$ and $x \in \varphi_{ji}^{-1}(\Omega_{ji} \cap \Omega_{jk})$, so by 3(c), we get $\varphi_{ji}^{-1}(\Omega_{ji} \cap \Omega_{jk}) \subseteq \Omega_{ik}$ and thus, $\varphi_{ki}(x)$ is defined and by 3(c) again,

$$\varphi_{ki}(x) = \varphi_{kj} \circ \varphi_{ji}(x) = z\,,$$

that is, $x \sim z$, as desired. Since $\sim$ is an equivalence relation let

$$M_{\mathcal{G}} = \left(\coprod_{i\in I} \Omega_i\right) / \sim$$

be the quotient set and let $p : \coprod_{i\in I} \Omega_i \to M_{\mathcal{G}}$ be the quotient map, with $p(x) = [x]$, where $[x]$ denotes the equivalence class of x (see Figure 4.3). Also, for every $i \in I$, let $\mathrm{in}_i : \Omega_i \to \coprod_{i\in I} \Omega_i$ be the natural injection and let

$$\tau_i = p \circ \mathrm{in}_i : \Omega_i \to M_{\mathcal{G}}\,.$$

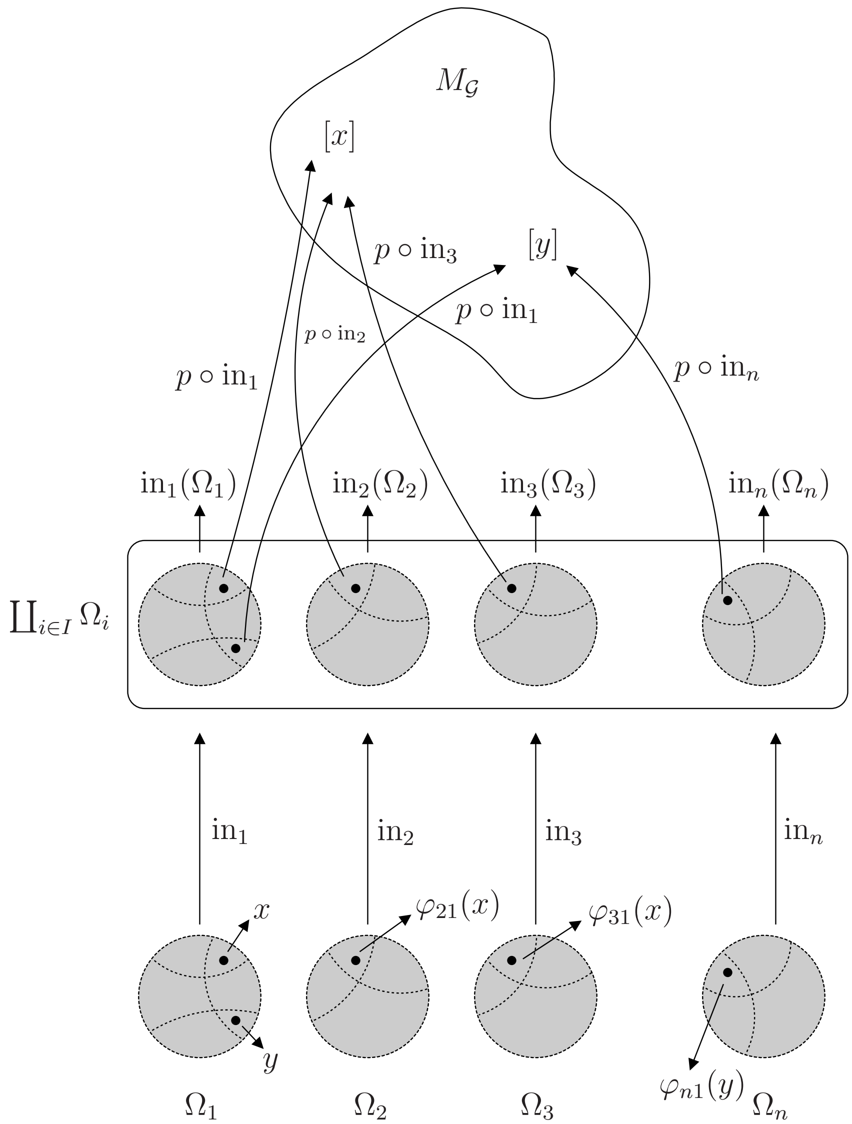

Figure 4.3: The quotient construction.

Since we already noted that if $x \sim y$ and $x, y \in \Omega_i$, then $x = y$, we conclude that every τ_i is injective. We give $M_{\mathcal{G}}$ the coarsest topology that makes the bijections, $\tau_i \colon \Omega_i \to \tau_i(\Omega_i)$, into homeomorphisms. Then, if we let $U_i = \tau_i(\Omega_i)$ and $\varphi_i = \tau_i^{-1}$, it is immediately verified that the (U_i, φ_i) are charts and this collection of charts forms a C^k atlas for $M_{\mathcal{G}}$. As there are countably many charts, $M_{\mathcal{G}}$ is second-countable. Therefore, for $M_{\mathcal{G}}$ to be a manifold it only remains to check that the topology is Hausdorff. For this, we use the following:

Claim. For all $(i, j) \in I \times I$, we have $\tau_i(\Omega_i) \cap \tau_j(\Omega_j) \neq \emptyset$ iff $(i, j) \in K$ and if so,

$$\tau_i(\Omega_i) \cap \tau_j(\Omega_j) = \tau_i(\Omega_{ij}) = \tau_j(\Omega_{ji})\,.$$

Proof of Claim. Assume that $\tau_i(\Omega_i) \cap \tau_j(\Omega_j) \neq \emptyset$ and let $[z] \in \tau_i(\Omega_i) \cap \tau_i(\Omega_j)$. Observe that $[z] \in \tau_i(\Omega_i) \cap \tau_i(\Omega_j)$ iff $z \sim x$ and $z \sim y$, for some $x \in \Omega_i$ and some $y \in \Omega_j$. Consequently, $x \sim y$, which implies that $(i, j) \in K$, $x \in \Omega_{ij}$ and $y \in \Omega_{ji}$. We have $[z] \in \tau_i(\Omega_{ij})$ iff $z \sim x$, for some $x \in \Omega_{ij}$. Then, either $i = j$ and $z = x$ or $i \neq j$ and $z \in \Omega_{ji}$, which shows that $[z] \in \tau_j(\Omega_{ji})$ and so,

$$\tau_i(\Omega_{ij}) \subseteq \tau_j(\Omega_{ji})\,.$$

Since the same argument applies by interchanging i and j, we have

$$\tau_i(\Omega_{ij}) = \tau_j(\Omega_{ji})\,,$$

for all $(i, j) \in K$. Since $\Omega_{ij} \subseteq \Omega_i$, $\Omega_{ji} \subseteq \Omega_j$, and $\tau_i(\Omega_{ij}) = \tau_j(\Omega_{ji})$, for all $(i, j) \in K$, we have

$$\tau_i(\Omega_{ij}) = \tau_j(\Omega_{ji}) \subseteq \tau_i(\Omega_i) \cap \tau_j(\Omega_j)\,,$$

for all $(i, j) \in K$. For the reverse inclusion, if $[z] \in \tau_i(\Omega_i) \cap \tau_j(\Omega_j)$, then we know that there is some $x \in \Omega_{ij}$ and some $y \in \Omega_{ji}$ such that $z \sim x$ and $z \sim y$, so $[z] = [x] \in \tau_i(\Omega_{ij})$ and $[z] = [y] \in \tau_j(\Omega_{ji})$, and then we get

$$\tau_i(\Omega_i) \cap \tau_j(\Omega_j) \subseteq \tau_i(\Omega_{ij}) = \tau_j(\Omega_{ji})\,.$$

This proves that if $\tau_i(\Omega_i) \cap \tau_j(\Omega_j) \neq \emptyset$, then $(i, j) \in K$ and

$$\tau_i(\Omega_i) \cap \tau_j(\Omega_j) = \tau_i(\Omega_{ij}) = \tau_j(\Omega_{ji})\,.$$

Finally, assume that $(i, j) \in K$. Then, for any $x \in \Omega_{ij} \subseteq \Omega_i$, we have $y = \varphi_{ji}(x) \in \Omega_{ji} \subseteq \Omega_j$ and $x \sim y$, so that $\tau_i(x) = \tau_j(y)$, which proves that $\tau_i(\Omega_i) \cap \tau_j(\Omega_j) \neq \emptyset$ and our claim is proved.
End of Proof of Claim.

We now prove that the topology of $M_{\mathcal{G}}$ is Hausdorff. Pick $[x], [y] \in M_{\mathcal{G}}$ with $[x] \neq [y]$, for some $x \in \Omega_i$ and some $y \in \Omega_j$. Either $\tau_i(\Omega_i) \cap \tau_j(\Omega_j) = \emptyset$, in which case, as τ_i and τ_j are homeomorphisms, $[x]$ and $[y]$ belong to the two disjoint open sets $\tau_i(\Omega_i)$ and $\tau_j(\Omega_j)$. If not, then by the Claim, $(i, j) \in K$ and

$$\tau_i(\Omega_i) \cap \tau_j(\Omega_j) = \tau_i(\Omega_{ij}) = \tau_j(\Omega_{ji})\,.$$

There are several cases to consider (refer to Figure 4.4):

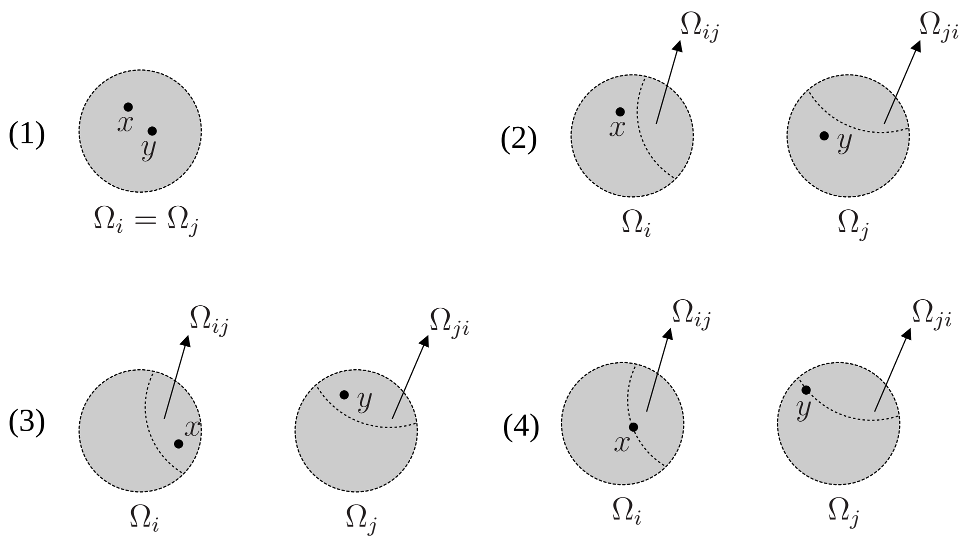

Figure 4.4: The four cases of the proof of Condition (4) of Definition 4.1.

(1) If $i = j$ then x and y can be separated by disjoint opens, V_x and V_y, and as τ_i is a homeomorphism, $[x]$ and $[y]$ are separated by the disjoint open subsets $\tau_i(V_x)$ and $\tau_j(V_y)$.

(2) If $i \neq j$, $x \in \Omega_i - \overline{\Omega_{ij}}$ and $y \in \Omega_j - \overline{\Omega_{ji}}$, then $\tau_i(\Omega_i - \overline{\Omega_{ij}})$ and $\tau_j(\Omega_j - \overline{\Omega_{ji}})$ are disjoint open subsets separating $[x]$ and $[y]$, where $\overline{\Omega_{ij}}$ and $\overline{\Omega_{ji}}$ are the closures of Ω_{ij} and Ω_{ji}, respectively.

(3) If $i \neq j$, $x \in \Omega_{ij}$ and $y \in \Omega_{ji}$, as $[x] \neq [y]$ and $y \sim \varphi_{ij}(y)$, then $x \neq \varphi_{ij}(y)$. We can separate x and $\varphi_{ij}(y)$ by disjoint open subsets, V_x and V_y, and $[x]$ and $[y] = [\varphi_{ij}(y)]$ are separated by the disjoint open subsets $\tau_i(V_x)$ and $\tau_i(V_y)$.

(4) If $i \neq j$, $x \in \partial(\Omega_{ij}) \cap \Omega_i$ and $y \in \partial(\Omega_{ji}) \cap \Omega_j$, then we use condition (4) of Definition 4.1. This condition yields two disjoint open subsets, V_x and V_y, with $x \in V_x$ and $y \in V_y$, such that no point of $V_x \cap \Omega_{ij}$ is equivalent to any point of $V_y \cap \Omega_{ji}$, and so $\tau_i(V_x)$ and $\tau_j(V_y)$ are disjoint open subsets separating $[x]$ and $[y]$.

Therefore, the topology of $M_{\mathcal{G}}$ is Hausdorff and $M_{\mathcal{G}}$ is indeed a manifold. Finally, it is trivial to verify that the transition functions of $M_{\mathcal{G}}$ are the original gluing functions, φ_{ij}. $\square$

The beauty of the idea of defining gluing data for constructing a manifold, M, is that it allows the construction of M without having prior knowledge of its topology (that is, without explicitly having the underlying topological space M). The construction is carried out by gluing open subsets of $\mathbb{R}^n$ (the Ω_i's) according to prescribed gluing instructions (namely, glue Ω_i and Ω_j by identifying

Ω_{ij} and Ω_{ji} using φ_{ji}). This way of specifying a manifold clearly separates the local structure of the manifold (given by the Ω_i's) from its global structure, which is specified by the gluing functions. Furthermore, the construction ensures that M is C^k (even for $k = \infty$) with no extra effort, as the gluing functions φ_{ji} are assumed to be C^k.

In [36, 54], a set of gluing data is called a *proto-manifold*. However, there are two subtle differences between our definition of gluing data and the definition of a proto-manifold in [36, 54]. First, the cocycle condition (condition 3(c)) of both definitions are slightly different, as the one used in the definition of a proto-manifold is not strong enough to imply transitivity of the relation $\sim$ in the proof of Theorem 4.1 (see Appendix A). Second, in the definition of a proto-manifold, there is no condition similar to condition 4 of Definition 4.1. However, in order to ensure that a Hausdorff manifold can always be constructed from a proto-manifold (in a way much like $M_{\mathcal{G}}$ is in Theorem 4.1), Grimm [54] requires that the quotient $(\Omega_i \coprod \Omega_j)/\sim$ be embeddable in $\mathbb{R}^n$ for all $(i,j) \in K$ with $i \neq j$. This requirement is stronger than condition 4 of Definition 4.1, and it prevents us from obtaining certain manifolds such as a 2-sphere resulting from gluing two open discs in $\mathbb{R}^2$ along an annulus (see [54], Appendix C).

4.2 Parametric Pseudo-Manifolds

It should be noted that as nice as it is, the proof of Theorem 4.1 gives us a theoretical construction, which yields an "abstract" manifold, $M_{\mathcal{G}}$, but does not yield any information on the geometry of this manifold. Furthermore, $M_{\mathcal{G}}$ may not be orientable or compact, even if we start with a finite set of p-domains. However, for the problem we are dealing with, we are given a simplicial surface and we want to build a "concrete" manifold: a surface in $\mathbb{R}^3$ that approximates the underlying surface of the simplicial surface. It turns out that it is always possible to define what we call a "pseudo-surface" from any given set of gluing data, which under certain conditions is a surface in $\mathbb{R}^3$, as we shall show later on in this section.

Definition 4.2. Let n, d, and k be three integers with $n > d \geq 1$ and $k \geq 1$ or $k = \infty$. A *parametric C^k pseudo-manifold of dimension d in $\mathbb{R}^n$* is a pair, $\mathcal{M} = (\mathcal{G}, (\theta_i)_{i\in I})$, such that $\mathcal{G} = \big((\Omega_i)_{i\in I}, (\Omega_{ij})_{(i,j)\in I\times I}, (\varphi_{ji})_{(i,j)\in K}\big)$ is a set of gluing data, for some finite set I, and each θ_i is a C^k function, $\theta_i : \Omega_i \to \mathbb{R}^n$, called a *parametrization* such that the following holds:

(C) For all $(i,j) \in K$, we have

$$\theta_i = \theta_j \circ \varphi_{ji}\,.$$

For short, we use the terminology *parametric pseudo-manifold*. The subset, $M \subset \mathbb{R}^n$, given by

$$M = \bigcup_{i\in I} \theta_i(\Omega_i)$$

is called the *image* of the parametric pseudo-manifold, $\mathcal{M}$. When $n = 3$ and $d = 2$, we say that $\mathcal{M}$ is a *parametric pseudo-surface*.

Condition (C) obviously implies that

$$\theta_i(\Omega_{ij}) = \theta_j(\Omega_{ji})\,,$$

for all $(i, j) \in K$. Consequently, θ_i and θ_j are consistent parametrizations of the overlap $\theta_i(\Omega_{ij}) = \theta_j(\Omega_{ij})$. Thus, the shape, M, whatever it is, is covered by pieces, $U_i = \theta_i(\Omega_i)$, not necessarily open, with each U_i parametrized by θ_i and where the overlapping pieces, $U_i \cap U_j$, are parametrized consistently. The local structure of M is given by the θ_i's and its global structure is given by the gluing data. More importantly, we can give M a manifold structure if we require the θ_i's to be bijective and to satisfy the following additional conditions:

(C') For all $(i, j) \in K$,

$$\theta_i(\Omega_i) \cap \theta_j(\Omega_j) = \theta_i(\Omega_{ij}) = \theta_j(\Omega_{ji})\,.$$

(C") For all $(i, j) \notin K$,

$$\theta_i(\Omega_i) \cap \theta_j(\Omega_j) = \emptyset\,.$$

If conditions (C') and (C") do not hold, we may not be able to give M a manifold structure. So, these conditions are actually necessary. Interestingly, regardless of the veracity of conditions (C') and (C"), we can still show that M is the image in $\mathbb{R}^n$ of the abstract manifold, $M_{\mathcal{G}}$, as stated by Proposition 4.2 below:

Proposition 4.2. Let $\mathcal{M} = (\mathcal{G}, (\theta_i)_{i\in I})$ be a parametric C^k pseudo-manifold of dimension d in $\mathbb{R}^n$, where $\mathcal{G} = \big((\Omega_i)_{i\in I}, (\Omega_{ij})_{(i,j)\in I\times I}, (\varphi_{ji})_{(i,j)\in K}\big)$ is a set of gluing data, for some finite set I. Then, the parametrization maps, θ_i, induce a surjective map, $\Theta \colon M_{\mathcal{G}} \to M$, from the abstract manifold, $M_{\mathcal{G}}$, specified by $\mathcal{G}$ to the image, $M \subseteq \mathbb{R}^n$, of the parametric pseudo-manifold, $\mathcal{M}$, and the following property holds: for every Ω_i, $\theta_i = \Theta \circ \tau_i$, where $\tau_i \colon \Omega_i \to M_{\mathcal{G}}$ are the parametrization maps of the manifold $M_{\mathcal{G}}$ (see the proof of Theorem 4.1 for the definition of τ_i).

Proof. Recall that

$$M_{\mathcal{G}} = \left(\coprod_{i\in I} \Omega_i\right) / \sim\,,$$

where $\sim$ is the equivalence relation defined so that, for all $x, y \in \coprod_{i\in I} \Omega_i$,

$$x \sim y \quad \text{iff} \quad (\exists (i, j) \in K)(x \in \Omega_{ij},\ y \in \Omega_{ji},\ y = \varphi_{ji}(x))\,.$$

The proof of Theorem 4.1 also showed that $\tau_i(\Omega_i) \cap \tau_j(\Omega_j) \neq \emptyset$ iff $(i, j) \in K$ and if so,

$$\tau_i(\Omega_i) \cap \tau_j(\Omega_j) = \tau_i(\Omega_{ij}) = \tau_j(\Omega_{ji})\,.$$

In particular,

$$\tau_i(\Omega_i - \Omega_{ij}) \cap \tau_j(\Omega_j - \Omega_{ji}) = \emptyset$$

for all $(i, j) \in I \times I$ ($\Omega_{ij} = \Omega_{ji} = \emptyset$ when $(i, j) \notin K$). These properties with the fact that the τ_i's are injections show that for all $(i, j) \notin K$, we can define $\Theta_i : \tau_i(\Omega_i) \to \mathbb{R}^n$ and $\Theta_j : \tau_j(\Omega_j) \to \mathbb{R}^n$ by

$$\Theta_i([x]) = \theta_i(x),\ x \in \Omega_i - \Omega_{ij} \quad \text{and} \quad \Theta_j([y]) = \theta_i(y),\ y \in \Omega_j - \Omega_{ji}\,.$$

It remains to define Θ_i on $\tau_i(\Omega_{ij})$ and Θ_j on $\tau_j(\Omega_{ji})$ in such a way that they agree on $\tau_i(\Omega_{ij}) = \tau_j(\Omega_{ji})$. However, condition (C) in Definition 4.2 says that for all $x \in \Omega_{ij}$,

$$\theta_i(x) = \theta_j(\varphi_{ji}(x))\,.$$

Consequently, if we define Θ_i on $\tau_i(\Omega_{ij})$ and Θ_j on $\tau_j(\Omega_{ji})$ by

$$\Theta_i([x]) = \theta_i(x),\ x \in \Omega_{ij} \quad \text{and} \quad \Theta_j([y]) = \theta_j(y),\ y \in \Omega_{ji}\,,$$

as $x \sim \varphi_{ji}(x)$, we have

$$\Theta_i([x]) = \theta_i(x) = \theta_j(\varphi_{ji}(x)) = \Theta_j([\varphi_{ji}(x)]) = \Theta_j([x])\,,$$

which means that Θ_i and Θ_j agree on $\tau_i(\Omega_{ij}) = \tau_j(\Omega_{ji})$. But then, the functions, Θ_i, agree whenever their domains overlap and consequently, they patch to yield a function, Θ, with domain $M_{\mathcal{G}}$ and image M, as desired. $\square$

From our discussion above, we have that the image, $M \subseteq \mathbb{R}^n$, of any parametric pseudo-manifold, $\mathcal{M} = (\mathcal{G}, (\theta_i)_{i \in I})$, defined from the same set of gluing data, $\mathcal{G}$, is the image of the abstract manifold, $M_{\mathcal{G}}$, in $\mathbb{R}^n$. So, the abstract manifold, $M_{\mathcal{G}}$, can be viewed as a "universal" manifold for the set $\mathcal{G}$. Moreover, whenever the θ_i's are bijective and conditions (C') and (C") hold, the subset M can be given the structure of a manifold.

4.3 Statement of the Problem

We are now ready to formalize the surface fitting problem we are dealing with: *given a simplicial surface, $\mathcal{K}$, in $\mathbb{R}^3$, a positive real number, ϵ, and a positive integer, k (or $k = \infty$), find a C^k surface, S, in $\mathbb{R}^3$ such that (1) S is homeomorphic to the underlying space, $|\mathcal{K}|$, of $\mathcal{K}$, and (2) there exists a homeomorphism, $h : |\mathcal{K}| \to S$, such that $\|p - h(p)\| \le \epsilon$, for every vertex p of $\mathcal{K}$.* Condition (1) requires the surfaces S and $|\mathcal{K}|$ be topologically equivalent, while condition (2) formalizes the requirement regarding the geometric proximity of S and the vertices of $\mathcal{K}$. We can view ϵ as an upper bound for the *approximation error* at the vertices of $\mathcal{K}$ with respect to h.

We solve the above problem by constructing a set of gluing data, $\mathcal{G}$, and a pseudo-parametric surface, $\mathcal{M} = (\mathcal{G}, (\theta_i)_{i \in I})$, from the given simplicial surface, $\mathcal{K}$, and its underlying space, $|\mathcal{K}|$, respectively. Our solution is a C^∞ surface, S, which is defined to be the image, M, of pseudo-parametric surface, $\mathcal{M}$. Unfortunately, our solution is not guaranteed to satisfy conditions (1) and (2). However, both conditions can in principle be enforced by a geometric procedure that checks for surface patch (self-)intersections and removes them by subdividing the input simplicial surface, $\mathcal{K}$. In this paper, we describe the construction of the set, $\mathcal{G}$, of gluing data (see Chapter 5). The construction of the parametrization functions and of the the pseudo-surface, $\mathcal{M}$, will be given in a subsequent paper.

Chapter 5

Building Sets of Gluing Data

This chapter describes a new construction to build a set of gluing data,

$$\mathcal{G} = \big((\Omega_i)_{i\in I}, (\Omega_{ij})_{(i,j)\in I\times I}, (\varphi_{ji})_{(i,j)\in K}\big)$$

from a given simplicial surface, $\mathcal{K}$, in $\mathbb{R}^3$. The triple $\mathcal{G}$ depends only on the topology of $\mathcal{K}$.

The task of designing gluing data that are at the same time, simple, efficiently implementable and theoretically correct, proved a lot more difficult that we expected. The main problem is to satisfy the cocycle condition 3(c) of Definition 4.1. We will spare the reader the history of our failed attempts and simply remark that the present solution is quite natural but that it took quite a bit of work to nail the details. A major technical breakthrough was the introduction of the *canonical lens* (see just after Definition 5.8). This long gestation is also reflected in the proofs of the propositions and lemmas in this chapter which are technically simple, but often long and tedious to follow. To make the chapter shorter and easier to read, we provide the more tedious proofs in Appendix A.

5.1 p-Domains, Gluing Domains, and Transition Functions

Let $\mathcal{K}$ be any given simplicial surface in $\mathbb{R}^3$, and let

$$\mathcal{G} = \big((\Omega_i)_{i\in I}, (\Omega_{ij})_{(i,j)\in I\times I}, (\varphi_{ji})_{(i,j)\in K}\big)$$

denote the set of gluing data we want to define. Hereafter, assume that the *degree* of every vertex v of K (i.e., the number of edges of $\mathcal{K}$ incident to v) is at least three. We now describe the construction of the set of p-domains, $(\Omega_i)_{i\in I}$, and the set of gluing domains, $(\Omega_{ij})_{(i,j)\in I\times I}$, of $\mathcal{G}$. Roughly speaking, each p-domain, Ω_i, in $(\Omega_i)_{i\in I}$ is the interior of a circle in $\mathbb{R}^2$; in turn, each gluing domain, Ω_{ij}, in $(\Omega_{ij})_{(i,j)\in I\times I}$ is defined by means of two abstractions, P-polygon and its canonical triangulation, and a composition of bijective functions.

Let

$$I = \{v \mid v \text{ is a vertex of } \mathcal{K}\}.$$

Definition 5.1. For every $v \in I$, the *p-domain* Ω_v is the set

$$\Omega_v = \left\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < \left(\cos\left(\frac{\pi}{m_v}\right)\right)^2\right\},$$

where m_v is the degree of vertex v.

Note that Ω_v is simply the interior of a circle of radius $\cos(\pi/m_v)$ centered at the origin of $\mathbb{R}^2$.

For any two $u, w \in I$, **we assume that Ω_u and Ω_w belong to distinct "copies" of $\mathbb{R}^2$.** This assumption ensures that $\Omega_u \cap \Omega_w = \emptyset$, so that condition (1) of Definition 4.1 holds. To build gluing domains and transition functions, we define the notions of a P-polygon and its canonical triangulation, as well as a bijective function that is a composition of two rotations around the origin, an analytic function, and a double reflection.

Definition 5.2. For each vertex v of $\mathcal{K}$, the *P-polygon, P_v, associated with* v is the regular polygon in $\mathbb{R}^2$ given by the vertices

$$v'_i = \left(\cos\left(\frac{2\pi \cdot i}{m_v}\right), \sin\left(\frac{2\pi \cdot i}{m_v}\right)\right),$$

for each $i \in \{0, \ldots, m_v - 1\}$, where m_v is the degree of of v.

Figure 5.1 illustrates Definition 5.2.

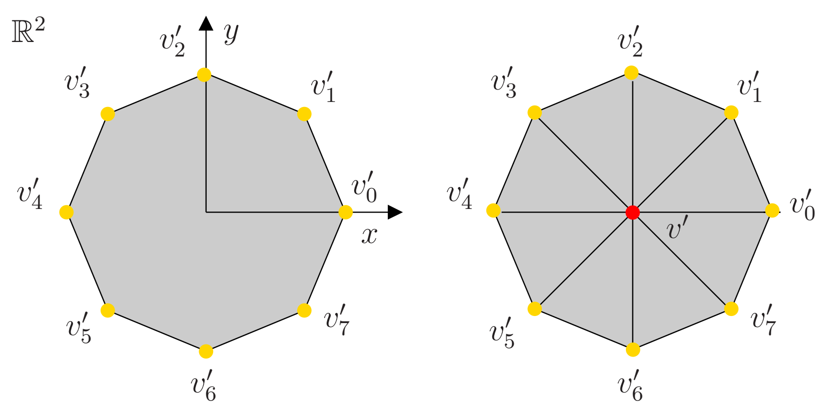

Figure 5.1: A P-polygon (left) and its canonical triangulation (right).

We assume that P_v resides in the copy of $\mathbb{R}^2$ that contains the p-domain Ω_v. So, Ω_v is the interior, $\text{int}(C_v)$, of the circle, C_v, inscribed in the P-polygon, P_v, i.e., $\Omega_v = \text{int}(C_v)$.

Definition 5.3. We can triangulate P_v by adding m_v diagonals and the vertex, $v' = (0,0)$, to P_v. Each diagonal connects v' to a vertex, v'_i, of P_u, for each $i = 0, \ldots, m_v - 1$. The resulting triangulation, denoted by T_v, is the *canonical triangulation of* P_v.

Figure 5.1 illustrates Definition 5.3.

Let v be any m-degree vertex in $\mathcal{K}$. Since $\mathcal{K}$ is a simplicial surface, the link, $lk(v, \mathcal{K})$, of v in $\mathcal{K}$ is homeomorphic to $\mathbb{S}^1$ (see Definition 3.6). So, $lk(v, \mathcal{K})$ is a simple, closed polygonal chain in $\mathbb{R}^3$. Let $v_0, \ldots, v_{m-1}$ be any enumeration of the vertices of $lk(v, \mathcal{K})$ such that $[v_i, v_{i+1}]$ is an edge of $lk(v, \mathcal{K})$, for each $i \in \{0, \ldots, m-1\}$, where the index $(i+1)$ should be always considered congruent modulo m (unless stated otherwise).

Definition 5.4. Given $st(v, \mathcal{K})$ and T_v, we define the function

$$s_v : st(v, \mathcal{K})^{((0))} \to T_v^{((0))}$$

such that $s_v(v) = v'$ and $s_v(v_i) = v'_i$, for every $i \in \{0, \ldots, m-1\}$. Note that for any $x, y, z \in st(v, \mathcal{K})$, we have that $[s_v(x), s_v(y)]$ is an edge of T_v if and only if $[x, y]$ is an edge of $st(v, \mathcal{K})$, and $[s_v(x), s_v(y), s_v(z)]$ is a triangle of T_v if and only if $[x, y, z]$ is a triangle of $st(v, \mathcal{K})$. This is to say that s_v is a simplicial isomorphism and that $st(v, \mathcal{K})$ and T_v are isomorphic. We can extend the bijection s_v to mapping triangles in $st(v, \mathcal{K})$ onto triangles in T_v. In particular, if $\sigma = [v, v_i, v_{i+1}]$ is in $st(v, \mathcal{K})$ then $s_v(\sigma) = [v', s_v(v_i), s_v(v_{i+1})]$ is its "image" in T_v.

Hereafter, we occasionally denote vertex $s_v(v)$ by v', for every $v \in st(v, \mathcal{K})$.

Definition 5.5. Let

$$\Pi : \mathbb{R}^2 - \{(0,0)\} \to (-\pi, \pi] \times \mathbb{R}_+$$

be the map that converts Cartesian to polar coordinates and is given by

$$\Pi(p) = \Pi((x, y)) = (\theta, r)\,,$$

for every $p \in \mathbb{R} - \{(0,0)\}$, where $\theta \in (-\pi, \pi]$ is the *angle* uniquely determined by

$$\cos\left(\frac{x}{r}\right) \quad \text{and} \quad \sin\left(\frac{y}{r}\right)\,,$$

and $r \in \mathbb{R}_+$ is the *length*, with

$$r = \sqrt{x^2 + y^2}\,.$$

Note that Π is bijective and its inverse,

$$\Pi^{-1} : (-\pi, \pi] \times \mathbb{R}_+ \to \mathbb{R}^2 - \{(0,0)\}\,,$$

is given by

$$\Pi^{-1}((\theta, r)) = (r \cdot \cos(\theta), r \cdot \sin(\theta))\,.$$

Note also that both Π and Π^{-1} are C^∞ functions. We use Π and Π^{-1} to define a map associated with each vertex of $\mathcal{K}$:

Definition 5.6. For each v in I and for each $p \in \mathbb{R}^2$, let

$$g_v : \mathbb{R}^2 - \{(0,0)\} \rightarrow \mathbb{R}^2 - \{(0,0)\}$$

be given by

$$g_v(p) = \Pi^{-1} \circ f_v \circ \Pi(p)$$

for every $p \in \mathbb{R}^2 - \{(0,0)\}$, where $f_v : (-\pi, \pi] \times \mathbb{R}_+ \rightarrow (-\pi, \pi] \times \mathbb{R}_+$ is given by

$$f_v((\theta, r)) = \left(\frac{m_v}{6} \cdot \theta, \frac{\cos(\pi/6)}{\cos(\pi/m_v)} \cdot r \right) ,$$

(θ, r) are the polar coordinates of p and m_v is the degree of vertex v in $\mathcal{K}$.

Function g_v has the following interpretation (refer to Figure 5.2): it maps the circular sector, A, of C_v onto the circular sector, B, of the circle of radius $\cos(\pi/6)$ and centers at $(0,0)$, where A consists of $(0,0)$ and all points with polar coordinates $(\theta, r) \in [-2\pi/m_v, 2\pi/m_v] \times (0, \cos(\pi/m_v)]$ and B consists of $(0,0)$ and all points with polar coordinates $(\beta, s) \in [-\pi/3, \pi/3] \times (0, \cos(\pi/6)]$. Note that A is contained in the quadrilateral given by the vertices v', $s_v(v_{m_v-1})$, $s_v(v_0)$, and $s_v(v_1)$ of T_v. We say that B is the *canonical sector*.

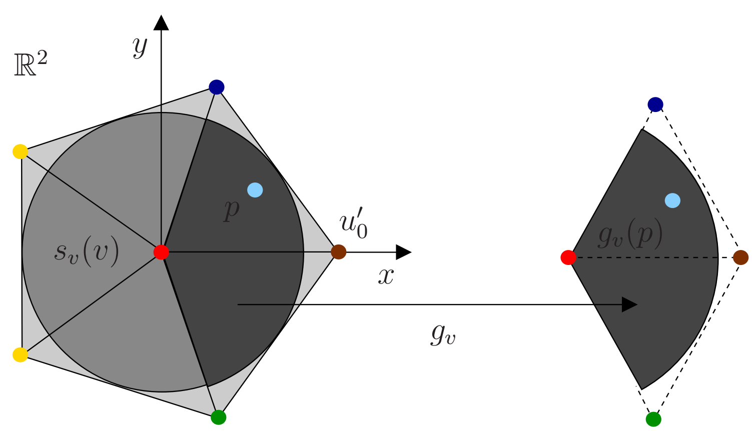

Figure 5.2: The action of g_v upon a point $p \in C_v$.

Function g_v is bijective and its inverse,

$$g_v^{-1} : \mathbb{R}^2 - \{(0,0)\} \rightarrow \mathbb{R}^2 - \{(0,0)\} ,$$

is given by

$$g_v^{-1}(q) = \Pi^{-1} \circ f_v^{-1} \circ \Pi(q)$$

for every $q \in \mathbb{R}^2 - \{(0,0)\}$, where $f_v^{-1} : (-\pi, \pi] \times \mathbb{R}_+ \rightarrow (-\pi, \pi] \times \mathbb{R}_+$ is given by

$$f_v^{-1}((\beta, s)) = \left(\frac{6}{m_v} \cdot \beta, \frac{\cos(\pi/m_v)}{\cos(\pi/6)} \cdot s \right) ,$$

(β, s) are the polar coordinates of q and m_v is the degree of vertex v in $\mathcal{K}$. Since f_v is clearly C^∞, so is g_v.

Definition 5.7. Let

$$h : \mathbb{R}^2 \to \mathbb{R}^2$$

be the function

$$h(p) = h((x, y)) = (1 - x, -y)\,,$$

for every point $p \in \mathbb{R}^2$ with rectangular coordinates (x, y).

Function h is a "double" reflection: $p = (x, y)$ is reflected over the line $x = 0.5$ and then over the line $y = 0$.

Definition 5.8. For any two u, w of I such that $[u, w]$ is an edge of $\mathcal{K}$, we define the function

$$g_{(u,w)} : \Omega_u - \{(0, 0)\} \to g_{(u,w)}(\Omega_u - \{(0, 0)\})$$

as

$$g_{(u,w)}(p) = R^{-1}_{(w,u)} \circ g^{-1}_w \circ h \circ g_u \circ R_{(u,w)}(p)$$

for every $p \in \Omega_u - \{(0, 0)\}$, where $R_{(u,w)}$ is a rotation around $(0, 0)$ that identifies the edge $[s_u(u) = u', s_u(w)]$ of T_u with its edge $[u', u'_0]$, and $R^{-1}_{(w,u)}$ is a rotation around $(0, 0)$ that identifies the edge $[s_w(w) = w', w'_0]$ of T_w with its edge $[w', w'_j]$, where $j \in \{0, 1, \ldots, m_w - 1\}$ and $s_w(u) = w'_j$.

Figure 5.3 shows the action of $g_{(u,w)}$ upon a point $p \in \Omega_u - \{(0, 0)\}$.

Note that $g_u \circ R_{(u,w)}$ maps $\Omega_u - \{(0, 0)\}$ onto the set $\text{int}(C) - \{(0, 0)\}$, where C is the circle of radius $\cos(\pi/6)$ and center $(0, 0)$ (see Figure 5.4). In turn, function h maps $\text{int}(C) - \{(0, 0)\}$ onto the set $\text{int}(D) - \{(1, 0)\}$, where D is the circle of radius $\cos(\pi/6)$ and center $(1, 0)$. Finally, by definition, the composite function $R^{-1}_{(w,u)} \circ g^{-1}_w$ maps $\text{int}(C) - \{(0, 0)\}$ onto $\Omega_w - \{(0, 0)\}$. So, only the points in $(\text{int}(C) - \{(0, 0)\}) \cap (\text{int}(D) - \{(1, 0)\})$ are mapped by $R^{-1}_{(w,u)} \circ g^{-1}_w$ to $\Omega_w - \{(0, 0)\}$. The set $E = (\text{int}(C) - \{(0, 0)\}) \cap (\text{int}(D) - \{(1, 0)\})$ is called the *canonical lens*, and it is contained in the quadrilateral, Q, given by the vertices $(0, 0)$, $(1/2, -\sqrt{3}/2)$, $(1, 0)$, and $(1/2, \sqrt{3}/2)$. Note that $\Omega_w - (0, 0)$ is *not* the image of $\text{int}(D) - \{(0, 0)\}$ by $R^{-1}_{w,u} \circ g^{-1}_w$, but the image of $\text{int}(C) - \{(0, 0)\}$.

Suppose that $[u, w, v]$ and $[u, w, z]$ are the two triangles of $\mathcal{K}$ sharing the edge $[u, w]$, where v and z are vertices of $\mathcal{K}$, with $v \neq z$. Let Q_u be the quadrilateral given by the vertices $s_u(u) = u'$, $s_u(v)$, $s_u(w)$, and $s_u(z)$. Then, the composite function $g_u \circ R_{(u,w)}$ maps the intersection $Q_u \cap (\Omega_u - \{(0, 0)\})$ onto the intersection set $Q \cap (\text{int}(C) - \{(0, 0)\})$. In turn, function h maps $Q \cap (\text{int}(C) - \{(0, 0)\})$ onto $Q \cap (\text{int}(D) - \{(0, 0)\})$. From the definition of h, the points in the upper (resp. lower) half of Q are mapped to the lower (resp. upper) half of Q. Next, the composite function $R^{-1}_{(w,u)} \circ g^{-1}_w$ maps the set $Q \cap (\text{int}(C) - \{(0, 0)\})$ onto the set $Q_w \cap (\Omega_w - \{(0, 0)\})$, where Q_w is the quadrilateral given by the vertices $s_w(w) = w'$, $s_w(z)$, $s_w(u)$, and $s_w(v)$. However, since only the points of $Q \cap (\text{int}(C) - \{(0, 0)\})$

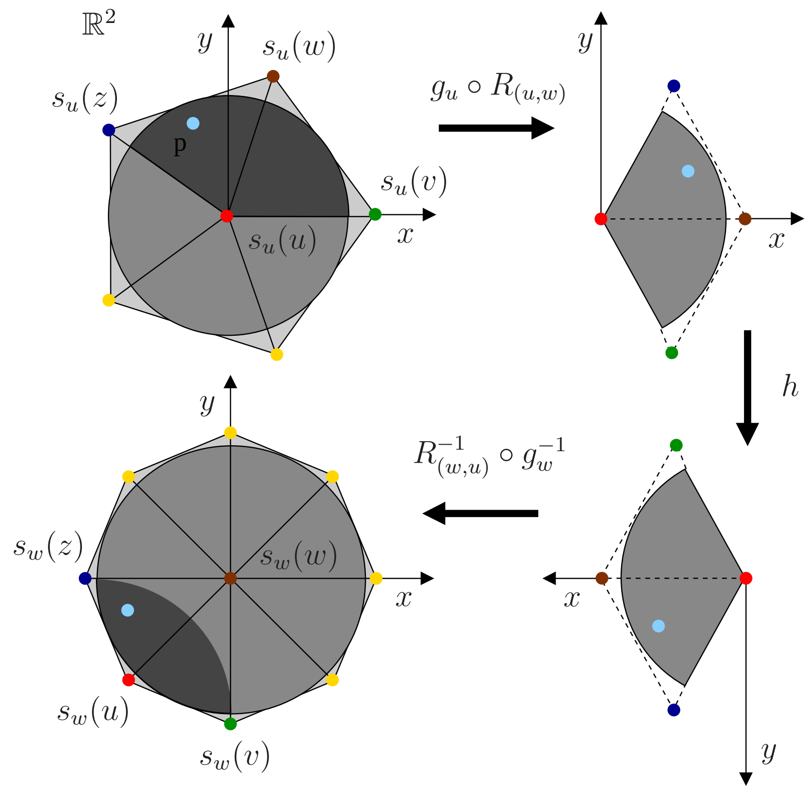

Figure 5.3: The action of $g_{(u,w)}$ upon a point $p \in \Omega_u - \{(0,0)\}$.

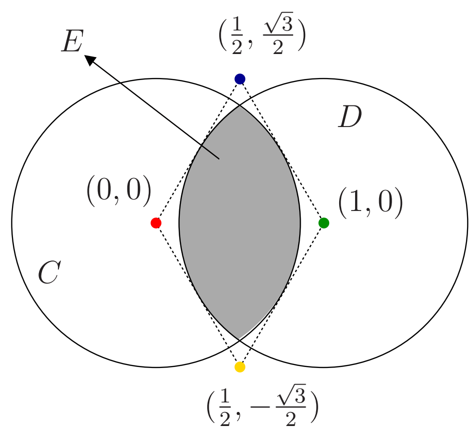

Figure 5.4: The circles C and D, the canonical lens E, and the quadrilateral Q (drawn with dotted line).

that belong to the canonical lens, E, are mapped by $R^{-1}_{(w,u)} \circ g^{-1}_{w}$ to $Q_w \cap (\Omega_w - \{(0,0)\})$, not all points of $Q_u \cap (\Omega_u - \{(0,0)\})$ get mapped by $g_{(u,w)}$ to $Q_w \cap (\Omega_w - \{(0,0)\})$. Finally, function $g_{(u,w)}$ is bijective and its inverse,

$$g^{-1}_{(u,w)} : g_{(u,w)}(\Omega_u - \{(0,0)\}) \to \Omega_u - \{(0,0)\}\,,$$

is given by

$$g^{-1}_{(u,w)}(q) = R^{-1}_{(u,w)} \circ g^{-1}_{u} \circ h \circ g_w \circ R_{(w,u)}(q)\,,$$

for every $q \in g_{(u,w)}(\Omega_u - \{(0,0)\})$.

The following propositions state several useful properties of $g_{(u,w)}$:

Proposition 5.1. For any two $u, w \in I$ such that $[u,w]$ is an edge of $\mathcal{K}$, function $g_{(u,w)}$ is C^∞.

Proof. By definition,

$$g_{(u,w)}(p) = R^{-1}_{(w,u)} \circ g^{-1}_{w} \circ h \circ g_u \circ R_{(u,w)}(p)\,,$$

for every $p \in \Omega_u - \{(0,0)\}$. Since $R^{-1}_{(w,u)}$, g^{-1}_{w}, h, g_u, and $R_{(u,w)}$ are all C^∞ functions, so is $g_{(u,w)}$. □

Proposition 5.2. For any two vertices, u and w, of $\mathcal{K}$ such that $[u,w]$ is an edge of $\mathcal{K}$, we have that $(0,0) \notin g_{(u,w)}(\Omega_u - \{(0,0)\})$.

Proof. If $[u,w]$ is an edge of $\mathcal{K}$ then $u \neq w$ and $g_{(u,w)}(p) = R^{-1}_{(w,u)} \circ g^{-1}_{w} \circ h \circ g_u \circ R_{(u,w)}(p)$, for every $p \in \Omega_u - \{(0,0)\}$. By definition of $g_u \circ R_{(u,w)}$, the point $q = g_u \circ R_{(u,w)}(p)$ is such that $\Pi(q) = (\theta, r)$, where $\theta \in (-\pi/3, \pi/3)$ and $r \in (0, \cos(\pi/6))$. So, the x coordinate of q is in the open interval $(0, \cos(\pi/6))$, which means that the x coordinate of $h(q)$ is in the open interval $(1 - \cos(\pi/6), 1)$. So, $h(q) \in \mathbb{R}^2 - \{(0,0)\}$. But, $R^{-1}_{(w,u)} \circ g^{-1}_{w}(\mathbb{R}^2 - \{(0,0)\}) = \mathbb{R}^2 - \{(0,0)\}$, and thus our claim is true. This is consistent with the fact that g^{-1}_{w} is undefined at $(0,0)$. □

Proposition 5.3. For any two vertices, u and w, of $\mathcal{K}$ such that $[u,w]$ is an edge of $\mathcal{K}$, we have that $g_{(u,w)}(\Omega_u - \{(0,0)\}) \cap (\Omega_w - \{(0,0)\})$ is non-empty and open in $\mathbb{R}^2$. Furthermore, $g^{-1}_{(u,w)} = g_{(w,u)}(p)$, for every p in $g_{(u,w)}(\Omega_u - \{(0,0)\}) \cap \Omega_w$.

Proof. By definition, we have that $g_{(u,w)}(p) = R^{-1}_{(w,u)} \circ g^{-1}_{w} \circ h \circ g_u \circ R_{(u,w)}(p)$, for every $p \in \Omega_u - \{(0,0)\}$. But, the composite function $h \circ g_u \circ R_{(u,w)}$ maps $\Omega_u - \{(0,0)\}$ onto the set $\operatorname{int}(D) - \{(1,0)\}$, where D is the circle of radius $\cos(\pi/6)$ and center $(1,0)$. In turn, the composite function $R^{-1}_{(w,u)} \circ g^{-1}_{w}$ maps $\operatorname{int}(C) - \{(0,0)\}$ onto $\Omega_w - \{(0,0)\}$, where C is the circle of radius $\cos(\pi/6)$ and center $(1,0)$. So, only the points of $\Omega_u - \{(0,0)\}$ that get mapped by $h \circ g_u \circ R_{(u,w)}$ to the canonical lens,

$$E = h \circ g_u \circ R_{(u,w)}(\Omega_u - \{(0,0)\}) \cap \operatorname{int}(C) - \{(0,0)\}\,,$$

are mapped by $R^{-1}_{(w,u)} \circ g^{-1}_w$ to $\Omega_w - \{(0,0)\}$. But, since the functions $R_{(u,w)}$, g_u, h, $R^{-1}_{(w,u)}$, and g^{-1}_w are all bijective and the canonical lens are non-empty, we have that $R^{-1}_{(w,u)} \circ g^{-1}_w(E)$ must be a non-empty subset of $\Omega_w - \{(0,0)\}$. So,

$$g_{(u,w)}(\Omega_u - \{(0,0)\}) \cap (\Omega_w - \{(0,0)\}) \neq \emptyset\,,$$

is true. To complete the proof of our first claim, we must show that the above set is open in $\mathbb{R}^2$. But, from Proposition 5.1, function $g_{(u,w)}$ is a homeomorphism. So, since the set $\Omega_u - \{(0,0)\}$ is open in $\mathbb{R}^2$, its image, $g_{(u,w)}(\Omega_u - \{(0,0)\})$, under $g_{(u,w)}$ is also open in $\mathbb{R}^2$. Because $\Omega_w - \{(0,0)\}$ is open in $\mathbb{R}^2$ and the intersection of open sets is again an open set, our claim follows.

Now, consider the second claim. By definition,

$$g^{-1}_{(u,w)}(p) = R^{-1}_{(u,w)}(p) \circ g^{-1}_u \circ h \circ g_w \circ R_{(w,u)}(p)\,,$$

for every $p \in g_{(u,w)}(\Omega_u - \{(0,0)\})$, and

$$g_{(w,u)}(q) = R^{-1}_{(u,w)}(p) \circ g^{-1}_u \circ h \circ g_w \circ R_{(w,u)}(q)\,,$$

for every $q \in \Omega_w - \{(0,0)\}$. So, $g^{-1}_{(u,w)}(t) = g_{(w,u)}(t)$, for every t in $g_{(u,w)}(\Omega_u - \{(0,0)\}) \cap (\Omega_w - \{(0,0)\})$. From Proposition 5.2,

$$(0,0) \notin g_{(u,w)}(\Omega_u - \{(0,0)\})\,.$$

So,

$$g_{(u,w)}(\Omega_u - \{(0,0)\}) \cap \Omega_w = g_{(u,w)}(\Omega_u - \{(0,0)\}) \cap (\Omega_w - \{(0,0)\})\,,$$

which implies that $g^{-1}_{(u,w)}(t) = g_{(w,u)}(t)$, for every t in $g_{(u,w)}(\Omega_u - \{(0,0)\}) \cap \Omega_w$, and thus our claim is true. $\square$

Function $g_{(u,w)}$ plays a crucial role in the following definitions of gluing domains and transition functions:

Definition 5.9. For any $u, w \in I$, the *gluing domain* Ω_{uw} is defined as

$$\Omega_{uw} = \begin{cases} \Omega_u & \text{if } u = w, \\ g_{(w,u)}(\Omega_w - \{(0,0)\}) \cap \Omega_u & \text{if } [u,w] \text{ is an edge of } \mathcal{K}, \\ \emptyset & \text{otherwise.} \end{cases}$$

As we shall see in Section 5.2, Definition 5.9 satisfies condition (2) of the definition of sets of gluing data (see Definition 4.1). Note that the requirement $\Omega_{uu} = \Omega_u$, for all $u \in I$, is true by definition. So, we are left to prove that Ω_{uw} is open in $\mathbb{R}^2$ and $\Omega_{uw} \neq \emptyset$ if and only if $\Omega_{wu} \neq \emptyset$, for each $(u,w) \in I \times I$, with $u \neq w$.

Transition functions are bijective maps between non-empty gluing domains defined as follows:

Definition 5.10. Let K be the index set,

$$K = \{(u, w) \in I \times I \mid \Omega_{uw} \neq 0\}.$$

Then, for any pair $(u, w) \in K$, the *transition function*,

$$\varphi_{wu} : \Omega_{uw} \to \Omega_{wu},$$

is such that, for every $p \in \Omega_{uw}$, we let

$$\varphi_{wu}(p) = \begin{cases} p & \text{if } u = w, \\ g_{(u,w)}(p) & \text{otherwise.} \end{cases}$$

Figure 5.5 illustrates Definition 5.10.

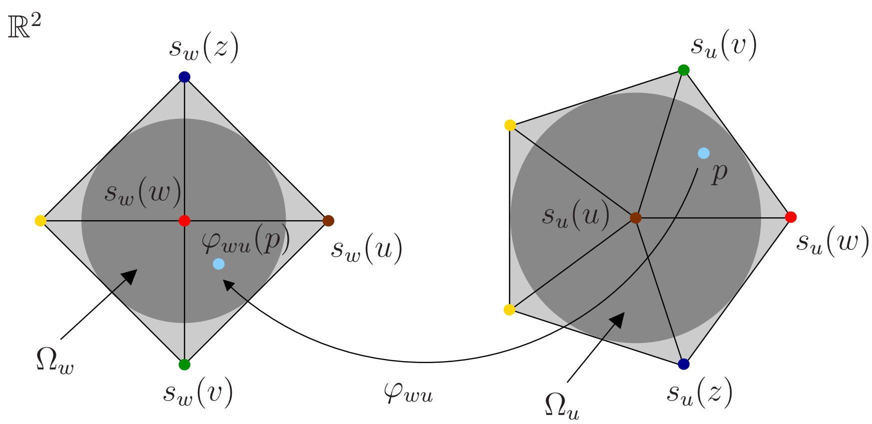

Figure 5.5: Illustration of Definition 5.10.

As we shall also see in Section 5.2, Definition 5.10 satisfies conditions (3) and (4) of the definition of sets of gluing data (see Definition 4.1). Note that condition 3(a), $\varphi_{uu} = \mathrm{id}_{\Omega_u}$, for all $u \in I$, is true by definition. So, we must prove condition 3(b), the *cocycle condition* (condition 3(c)), and the *Hausdorff condition* (condition (4)).

5.2 Construction Correctness

Propositions 5.4 and 5.5 below imply that Definition 5.9 satisfies condition (2) of Definition 4.1:

Proposition 5.4. Let Ω_u and Ω_w be any two p-domains of $(\Omega_v)_{v \in I}$. Then, $\Omega_{uw} \neq \emptyset$ if and only if $\Omega_{wu} \neq \emptyset$.

Proof. If $u = w$, our claim is trivially true. So, let us assume that $u \neq w$. Now, suppose that $\Omega_{uw} \neq \emptyset$. So, from Definition 5.9, we must have that $[u, w]$ is an edge of $\mathcal{K}$. Otherwise, Ω_{uw} would be empty. This implies that $g_{(u,w)}$ and its inverse, $g^{-1}_{(u,w)}$, are well-defined. Furthermore, Ω_{uw} and Ω_{wu} are defined as follows:

$$\Omega_{uw} = g_{(w,u)}(\Omega_w - \{(0,0)\}) \cap \Omega_u$$

and

$$\Omega_{wu} = g_{(u,w)}(\Omega_u - \{(0,0)\}) \cap \Omega_w \,.$$

From Proposition 5.2, we know that $(0,0) \notin g_{(w,u)}(\Omega_w - \{(0,0)\})$. So,

$$\Omega_{uw} = g_{(w,u)}(\Omega_w - \{(0,0)\}) \cap (\Omega_u - \{(0,0)\}) \,.$$

From Proposition 5.3, we know that $g_{(u,w)}$ and $g^{-1}_{(w,u)}$ coincide in Ω_{uw}. So,

$$g_{(u,w)}(\Omega_{uw}) = g^{-1}_{(w,u)}(\Omega_{uw}) = g^{-1}_{(w,u)}(g_{(w,u)}(\Omega_w - \{(0,0)\}) \cap (\Omega_u - \{(0,0)\})) \,.$$

Since $g^{-1}_{(w,u)}$ is bijective, we have that

$$\begin{aligned} g^{-1}_{(w,u)}(g_{(w,u)}(\Omega_w - \{(0,0)\}) \cap \Omega_u - \{(0,0)\})) &= g^{-1}_{(w,u)}(g_{(w,u)}(\Omega_w - \{(0,0)\})) \cap g^{-1}_{(w,u)}(\Omega_u - \{(0,0)\}) \\ &= (\Omega_w - \{(0,0)\}) \cap g^{-1}_{(w,u)}(\Omega_u - \{(0,0)\}) \\ &= g_{(u,w)}(\Omega_u - \{(0,0)\}) \cap (\Omega_w - \{(0,0)\}) \\ &= g_{(u,w)}(\Omega_u - \{(0,0)\}) \cap \Omega_w \\ &= \Omega_{wu} \,. \end{aligned}$$

Since $\Omega_{uw} \neq \emptyset$ and $g_{(u,w)}$ is bijective, the set $\Omega_{wu} = g_{(u,w)}(\Omega_{uw})$ cannot be empty either, and hence our claim follows. □

Proposition 5.5. Let Ω_u and Ω_w be any two p-domains of $(\Omega_v)_{v\in I}$. Then, the gluing domain Ω_{uw} is an open set of $\mathbb{R}^2$.

Proof. If $u = w$ then our claim is trivially true, as $\Omega_{uu} = \Omega_u$ and Ω_u is open in $\mathbb{R}^2$ (by definition). So, assume that $u \neq w$. If $\Omega_{uw} = \emptyset$ then our claim is trivially true. So, assume that $\Omega_{uw} \neq \emptyset$. From Definition 5.9, if $\Omega_{uw} \neq \emptyset$ then

$$\Omega_{uw} = g_{(w,u)}(\Omega_w - \{(0,0)\}) \cap \Omega_u \,.$$

From Proposition 5.2, we know that $(0,0) \notin g_{(u,w)}(\Omega_u - \{(0,0)\})$. So,

$$\Omega_{uw} = g_{(w,u)}(\Omega_w - \{(0,0)\}) \cap (\Omega_u - \{(0,0)\}) \,.$$

Finally, Proposition 5.3 states that the above set is non-empty and open in $\mathbb{R}^2$. □

In what follows, we show that the transition functions, as defined before, satisfy conditions (3) and (4) of Definition 4.1. Although conditions (3)(a) and (3)(b) follow from Condition (3)(c), the exposition of our proof of Condition (3)(c) assumes that (3)(a) and 3(b) are true, so we first show that condition (3)(b) holds.

Proposition 5.6. For any $(u, w) \in K$, we have that $\varphi_{wu}(p) = \varphi_{uw}^{-1}(p)$, for all $p \in \Omega_{uw}$.

Proof. From Definition 5.10, if $u = w$ then $\varphi_{wu} = \varphi_{uw} = \mathrm{id}_{\Omega_u}$. Otherwise, we have $\varphi_{wu} = g_{(u,w)}$ and $\varphi_{uw} = g_{(w,u)}$. In the former case, our claim is trivially true. In the latter case, Proposition 5.3 states that $g^{-1}_{(u,w)}(p) = g_{(w,u)}(p)$, for every $p \in \Omega_{uw}$. Since $\varphi_{uw}(p) = g_{(w,u)}(p) = g^{-1}_{(u,w)}(p) = \varphi^{-1}_{uw}(p)$, our claim follows. □

Our proof of Condition 3(c) relies on a property of function g_u, called *rotational symmetry*, which is stated below:

Proposition 5.7. Let $[u, w, z]$ be any triangle of $\mathcal{K}$. If $s_u(z)$ precedes $s_u(w)$ in a counterclockwise traversal of the vertices of P_u, then

$$M_{-\pi/3} \circ g_u \circ R_{(u,w)}(\Omega_{uw}) = g_u \circ R_{(u,w)}(\Omega_{uz}) \quad \text{and} \quad M_{\pi/3} \circ g_u \circ R_{(u,z)}(\Omega_{uz}) = g_u \circ R_{(u,z)}(\Omega_{uw})\,,$$

where $M_{-\frac{\pi}{3}}$ (resp. $M_{\frac{\pi}{3}}$) is a rotation by $-\frac{\pi}{3}$ (resp. $\frac{\pi}{3}$) around the origin. Furthermore,

$$\Omega_{uz} = M_{-\frac{2\pi}{m_u}}(\Omega_{uw}) \qquad \text{and} \qquad \Omega_{uw} = M_{\frac{2\pi}{m_u}}(\Omega_{uz})\,,$$

where $M_{-\frac{2\pi}{m_u}}$ is a rotation by $-\frac{2\pi}{m_u}$ around the origin, and m_u is the degree of vertex u in $\mathcal{K}$.

Proof. See Appendix A for a proof. □

We now show that the first implication of Condition 3(c) of Definition 4.1 holds:

Lemma 5.8. Let Ω_u, Ω_w, and Ω_x be any three p-domains in $(\Omega_v)_{v \in I}$. If the intersection

$$\Omega_{xu} \cap \Omega_{xw}$$

is nonempty, then

$$\varphi^{-1}_{xu}(\Omega_{xu} \cap \Omega_{xw}) \subseteq \Omega_{uw}\,.$$

Proof. See Appendix A for a proof. □

In what follows we show that the second and last implication of Condition 3(c) of Definition 4.1 also holds:

Lemma 5.9. Let Ω_u, Ω_w, and Ω_x be any three p-domains in $(\Omega_v)_{v \in I}$. If $\Omega_{xu} \cap \Omega_{xw} \neq \emptyset$, then

$$\varphi_{wu}(p) = \varphi_{wx} \circ \varphi_{xu}(p)\,,$$

for all $p \in \varphi^{-1}_{xu}(\Omega_{xu} \cap \Omega_{xw}) \subseteq \Omega_{uw}$.

Proof. See Appendix A for a proof. □

Lemma 5.10. Let (u, w) be any pair in K, with $u \neq w$. Then, for every $x \in \partial(\Omega_{uw}) \cap \Omega_u$ and every $y \in \partial(\Omega_{wu}) \cap \Omega_w$, there are open balls, V_x and V_y, centered at x and y, such that no point of $V_y \cap \Omega_{wu}$ is the image of any point $V_x \cap \Omega_{uw}$ under φ_{wu}.

Proof. By definition, each gluing domain, Ω_{uw}, is the image by $R^{-1}_{(u,w)} \circ g_u^{-1}$ of the canonical lens, E, given by

$$(\mathrm{int}(C) - \{(0,0)\}) \cap (\mathrm{int}(D) - \{(1,0)\}) ,$$

where C and D are the circles of radius $\cos(\pi/6)$ and centers $(0,0)$ and $(1,0)$, respectively. Furthermore, the gluing domain Ω_{uw} is also a lens-shaped set whose boundary, $\partial(\Omega_{uw})$, is the image by $R^{-1}_{(u,w)} \circ g_u^{-1}$ of the boundary, $\partial(E)$, of E. We can view $\partial(\Omega_{uw})$ as the union of two open and simple curve segments, C_{u_e} and C_{u_i}, such that C_{u_e} belongs to $\partial(\Omega_{uw})$ and the interior, $\mathrm{int}(C_{u_i})$, of C_{u_i} belongs to the interior of Ω_u, as shown in Figure 5.6. In addition, the pairs of endpoints of both curves, C_{u_e} and C_{u_i}, are the same, and each pair is the image by $R^{-1}_{(u,w)} \circ g_u^{-1}$ of the two intersection points of the boundaries, $\partial(C)$ and $\partial(D)$, of C and D.

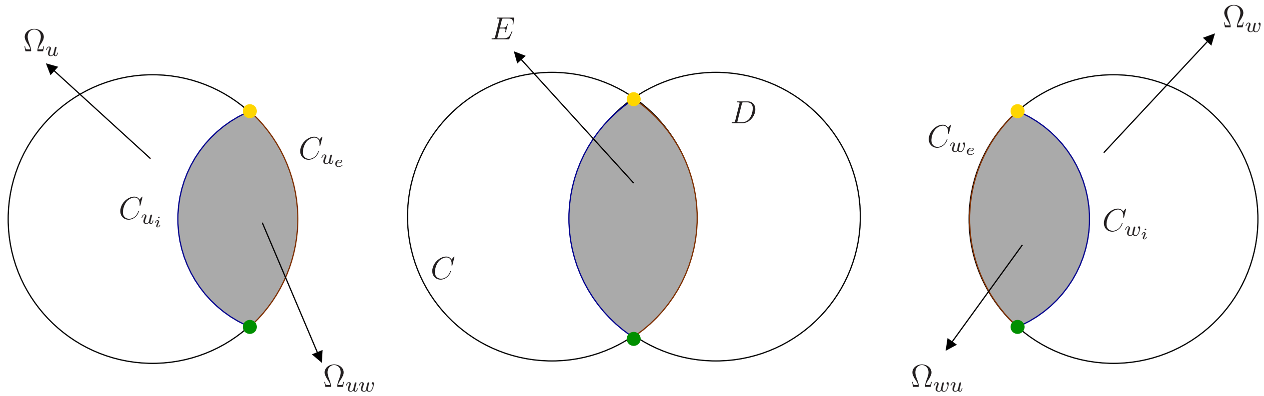

Figure 5.6: The image sets of the canonical lens, E, under $R^{-1}_{(u,w)} \circ g_u^{-1}$ and $R^{-1}_{(w,u)} \circ g_w^{-1}$.

Similarly, the boundary, $\partial(\Omega_{wu})$, of the gluing domain, Ω_{wu}, can be viewed as the union of two curves, C_{w_e} and C_{w_i}, such that C_{w_e} belongs to $\partial(\Omega_{wu})$ and the interior, $\mathrm{int}(C_{w_i})$, of C_{w_i} belongs to the interior of Ω_w. In addition, the pairs of endpoints of both curves, C_{w_e} and C_{w_i}, are the same, and each pair is the image by $R^{-1}_{(w,u)} \circ g_w^{-1}$ of the two intersection points of the boundaries, $\partial(C)$ and $\partial(D)$, of C and D (see Figure 5.6).

Note that

$$\mathrm{int}(C_{u_i}) = \partial(\Omega_{uw}) \cap \Omega_u \qquad \text{and} \qquad \mathrm{int}(C_{w_i}) = \partial(\Omega_{wu}) \cap \Omega_w .$$

Note also that

$$g_{(u,w)}(C_{u_i}) = C_{w_e} \qquad \text{and} \qquad g_{(w,u)}(C_{w_i}) = C_{u_e} .$$

Indeed,

$$g_{(u,w)}(C_{u_i}) = R^{-1}_{(w,u)} \circ g_w^{-1} \circ h \circ g_u \circ R_{(u,w)}(C_{u_i})\,.$$

By construction, we know that $g_u \circ R_{(u,w)}(C_{u_i}) \in \partial(C)$, which means that $h \circ g_u \circ R_{(u,w)}(C_{u_i}) \in \partial(D)$. So, we get

$$R^{-1}_{(w,u)} \circ g_w^{-1} \circ h \circ g_u \circ R_{(u,w)}(C_{u_i}) = C_{w_e}\,.$$

Finally, let x be any point in $\partial(\Omega_{uw}) \cap \Omega_u$. Since $\mathrm{int}(C_{u_i}) = \partial(\Omega_{uw}) \cap \Omega_u$, we have that $x \in \mathrm{int}(C_{u_i})$. From our discussion above, we also have that if $p = g_{(u,w)}(x)$ then $p \in \mathrm{int}(C_{w_e})$. Since $\mathrm{int}(C_{w_e}) \cap \mathrm{int}(C_{w_i}) = \emptyset$, there exists an open ball, V_p, centered at p such that $V_p \cap \mathrm{int}(C_{w_i}) = \emptyset$, which follows from the fact that $\mathbb{R}^2$ is a Hausdorff space.

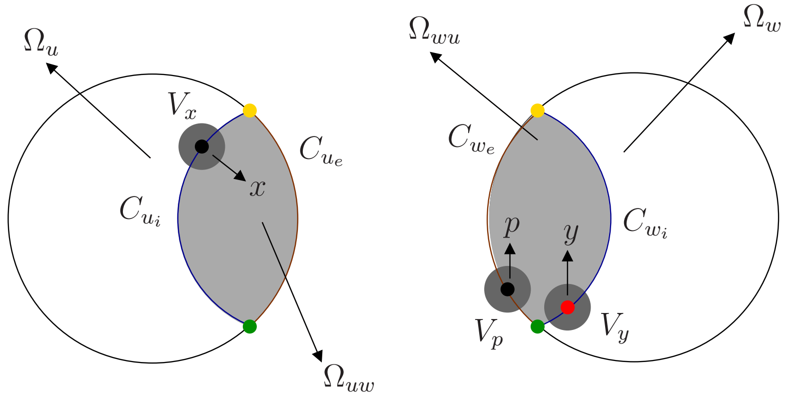

Figure 5.7: The open balls V_x, V_y, and V_p.

Since $\mathrm{int}(C_{w_i}) = \partial(\Omega_{wu}) \cap \Omega_w$, we get that

$$V_p \cap (\partial(\Omega_{wu}) \cap \Omega_w) = \emptyset\,.$$

In turn, for any point $y \in \partial(\Omega_{wu}) \cap \Omega_w$, there exists an open ball, V_y, such that $V_y \cap V_p = \emptyset$ (see Figure 5.7). This also follows from the fact that $\mathbb{R}^2$ is a Hausdorff space. So, define V_x to be any open ball centered at x such that $V_x \subseteq g^{-1}_{(u,w)}(V_p)$. By construction, we know that $g_{(u,w)}(V_x) \cap V_y = \emptyset$. To conclude that our claim is true, it suffices to notice that $g_{(u,w)}(V_x \cap \Omega_{uw}) \subset \Omega_w$ and that $\varphi_{wu} = g_{(u,w)}$ for every point in Ω_{uw}, which implies that

$$\varphi_{wu}(V_x \cap \Omega_{uw}) \cap (V_y \cap \Omega_{wu}) = \emptyset\,.$$

$\square$

The following theorem states the correctness of the construction in Section 5.1:

Theorem 5.11. Given any given simplicial surface, $\mathcal{K}$, in $\mathbb{R}^3$, the triple

$$\mathcal{G} = ((\Omega_v)_{v \in I}, (\Omega_{uw})_{(u,w) \in I \times I}, (\varphi_{uw})_{(u,w) \in K})\,,$$

where

- $(\Omega_v)_{v \in I}$ is any set of p-domains for $\mathcal{K}$,
- $(\Omega_{uw})_{(u,w) \in I \times I}$ is the set of gluing domains for $\mathcal{K}$ with respect to $(\Omega_v)_{v \in I}$,
- $(\varphi_{uw})_{(u,w) \in K}$ is the set of transition functions defined by Definition 5.10, and
- $K = \{(u, w) \in I \times I \mid \Omega_{uw} \neq \emptyset\}$,

is a set of gluing data according to Definition 4.1.

Proof. Our claim follows immediately from the facts that our construction yields p-domains, gluing domains, and transition functions that satisfy conditions (1)-(4) of the definition of a set of gluing data (see Definition 5.10). Indeed, the p-domains are open sets in $\mathbb{R}^2$; Proposition 5.4 and Proposition 5.5 ensure that the gluing domains satisfy condition (2) of Definition 5.10; Proposition 5.6, Lemma 5.8, and Lemma 5.9 ensure that the transition functions satisfy condition (3); and Lemma 5.10 states that condition (4) also hold. $\square$

From now on, we shall refer to

$$\mathcal{G} = ((\Omega_v)_{v \in I}, (\Omega_{uw})_{(u,w) \in I \times I}, (\varphi_{uw})_{(u,w) \in K})$$

as a *set of gluing data for* $\mathcal{K}$.

Finally, we show that the transition functions are all C^∞ functions:

Lemma 5.12. For any pair $(u, w) \in K$, the transition function φ_{wu} is C^∞.

Proof. From Definition 5.10, we know that φ_{wu} is the identity function if $u = w$ and the function $g_{(u,w)}$ otherwise. In the former case, our claim is trivially true. In the latter case, our claim follows from Proposition 5.1. $\square$

Chapter 6

Conclusion and Future Work

6.1 Conclusion

We have presented the mathematical framework for a new constructive solution to the problem of fitting a smooth surface to a given simplicial surface. Our construction is based on the manifold-based approach pioneered by Grimm and Hughes [36, 54]. The key idea behind this approach is to define a surface by overlapping surface patches via a gluing process, as opposed to stitching them together along their common boundary curves.

Like the manifold-based constructions in [36, 39], ours has also been devised for simplicial surfaces, which are far more popular than quadrilateral surfaces in computer graphics and geometry processing applications [59]. In addition, our construction is more compact and simpler than the one in [36], more powerful than the construction in [39] (as the surfaces generated by our construction do not contain singularities), and shares with [38], a construction devised for quadrilateral surfaces, the ability of producing C^∞-continuous surfaces and the flexibility in ways of defining the geometry of the resulting surface.

Our framework improves upon the one given in [54] (which has been used to undergird the constructions given in [36, 37, 38]) in three ways:

(1) We give a corrected version of the *cocycle condition* (condition 3(c) in Definition 4.1) which ensures the transitivity of the equivalence relation, $\sim$, used in Theorem 4.1.

(2) We give a more general criterion (condition (4) in Definition 4.1) ensuring that the quotient manifold, $M_{\mathcal{G}}$, of Theorem 4.1 is Hausdorff.

(3) We give a more general and simpler construction of concrete sets of gluing data (see Chapter 5).

We also introduce the notion of parametric pseudo-manifold, which is a concrete object that can actually be constructed from gluing data, as opposed to the abstract quotient manifold, $M_{\mathcal{G}}$. We also show that any parametric pseudo-manifold, $\mathcal{M}$, constructed from a set of gluing data, $\mathcal{G}$, is the image of the abstract quotient manifold, $M_{\mathcal{G}}$ (Proposition 4.2). In a sequel to this paper, we will describe a new method for constructing parameter functions defining a pseudo-surface approximating a given triangular mesh.

6.2 On-going and Future Work

There are two immediate extensions of the work presented here, namely:

- surface construction from very large triangle meshes,
- the incorporation of sharp features and meshes wih boundaries.

The construction of smooth surfaces from very large meshes (i.e., simplicial surfaces with hundreds of thousands or millions of triangles) has already been studied before (see [61, 62, 63, 64], to name a few). In particular, an extension of the manifold-based construction in [39] to fit smooth surfaces to very large simplicial surfaces is described in [65]. Here, the goal is to define surface patches that cover regions of the input surface containing several small triangles, as opposed to only one triangle or the star of a vertex. By doing so, it is possible to obtain a reasonably small smooth surface representation for the input surface. Currently, we are developing an extension of our manifold-based construction to deal with very large simplicial surfaces.

Although the manifold-based approach is meant to be used to construct smooth surfaces, there are several 3D shapes whose boundary is a smooth surface everywhere, but along certain curves and corners known as *sharp features*. For modeling such boundaries, it would be appropriate to apply a manifold-based construction that is capable of generating C^k-continuous surfaces where $k = 0$ along sharp features and $k > 0$ or $k = \infty$ everywhere else. Sharp features can be extracted from the input simplicial surface, $\mathcal{K}$, using existing tools for feature detection on triangle meshes [66]. Next, we map the features from $\mathcal{K}$ to the p-domains. Finally, we define shape functions that are not smooth at points and lines of the p-domains corresponding to the features. Currently, we are investigating the details of the last two steps of this approach.

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Appendix A

Proofs and Counterexamples

A.1 Proofs

Proposition 5.7. Let $[u, w, z]$ be any triangle of $\mathcal{K}$. If $s_u(z)$ precedes $s_u(w)$ in a counterclockwise traversal of the vertices of P_u, then

$$M_{-\pi/3} \circ g_u \circ R_{(u,w)}(\Omega_{uw}) = g_u \circ R_{(u,w)}(\Omega_{uz}) \quad \text{and} \quad M_{\pi/3} \circ g_u \circ R_{(u,z)}(\Omega_{uz}) = g_u \circ R_{(u,z)}(\Omega_{uw}) \,,$$

where $M_{-\frac{\pi}{3}}$ (resp. $M_{\frac{\pi}{3}}$) is a rotation by $-\frac{\pi}{3}$ (resp. $\frac{\pi}{3}$) around the origin. Furthermore,

$$\Omega_{uz} = M_{-\frac{2\pi}{m_u}}(\Omega_{uw}) \qquad \text{and} \qquad \Omega_{uw} = M_{\frac{2\pi}{m_u}}(\Omega_{uz}) \,,$$

where $M_{-\frac{2\pi}{m_u}}$ is a rotation by $-\frac{2\pi}{m_u}$ around the origin, and m_u is the degree of vertex u in $\mathcal{K}$.

Proof. From Definition 5.9, we have that

$$\Omega_{uw} = g_{(w,u)}(\Omega_w - \{(0,0)\}) \cap \Omega_u \quad \text{and} \quad \Omega_{uz} = g_{(z,u)}(\Omega_z - \{(0,0)\}) \cap \Omega_u \,.$$

From Proposition 5.2, we know that $(0,0) \notin g_{(w,u)}(\Omega_w - \{(0,0)\})$ and $(0,0) \notin g_{(z,u)}(\Omega_z - \{(0,0)\})$. So,

$$\Omega_{uw} = g_{(w,u)}(\Omega_w - \{(0,0)\}) \cap (\Omega_u - \{(0,0)\}) \quad \text{and} \quad \Omega_{uz} = g_{(z,u)}(\Omega_z - \{(0,0)\}) \cap (\Omega_u - \{(0,0)\}) \,.$$

Since $g_u \circ R_{(u,w)}$ and $g_u \circ R_{(u,z)}$ are bijective, we also have that

$$\begin{aligned} g_u \circ R_{(u,w)}(g_{(w,u)}(\Omega_w - \{(0,0)\}) \cap (\Omega_u - \{(0,0)\})) &= g_u \circ R_{(u,w)}(g_{(w,u)}(\Omega_w - \{(0,0)\})) \\ &\quad \cap g_u \circ R_{(u,w)}(\Omega_u - \{(0,0)\}) \end{aligned}$$

and

$$g_u \circ R_{(u,z)}(g_{(z,u)}(\Omega_z - \{(0,0)\}) \cap (\Omega_u - \{(0,0)\})) = g_u \circ R_{(u,z)}(g_{(z,u)}(\Omega_z - \{(0,0)\})) \\ \cap\, g_u \circ R_{(u,z)}(\Omega_u - \{(0,0)\})\,.$$

But,

$$g_u \circ R_{(u,w)}(\Omega_u - \{(0,0)\}) = \mathrm{int}(C) - \{(0,0)\} \quad \text{and} \quad g_u \circ R_{(u,z)}(\Omega_u - \{(0,0)\}) = \mathrm{int}(C) - \{(0,0)\}\,,$$

where C is the circle of radius $\cos(\pi/6)$ and center $(0,0)$,

$$\begin{aligned} g_u \circ R_{(u,w)}(g_{(w,u)}(\Omega_w - \{(0,0)\})) &= g_u \circ R_{(u,w)} \circ R^{-1}_{(u,w)} \circ g_u^{-1} \circ h \circ g_w \circ R_{(w,u)}(\Omega_w - \{(0,0)\}) \\ &= h \circ g_w \circ R_{(w,u)}(\Omega_w - \{(0,0)\}) \\ &= \mathrm{int}(D) - \{(1,0)\}\,, \end{aligned}$$

where D is the circle of radius $\cos(\pi/6)$ and center $(1,0)$, and

$$\begin{aligned} g_u \circ R_{(u,w)}(g_{(z,u)}(\Omega_z - \{(0,0)\})) &= g_u \circ R_{(u,w)} \circ R^{-1}_{(u,z)} \circ g_u^{-1} \circ h \circ g_z \circ R_{(z,u)}(\Omega_w - \{(0,0)\}) \\ &= g_u \circ M_{-\frac{2\pi}{m_u}} \circ g_u^{-1} \circ h \circ g_w \circ R_{(w,u)}(\Omega_w - \{(0,0)\}) \\ &= M_{-\frac{\pi}{3}} \circ h \circ g_w \circ R_{(w,u)}(\Omega_w - \{(0,0)\}) \\ &= M_{-\frac{\pi}{3}}(\mathrm{int}(D) - \{(1,0)\}) \\ &= \mathrm{int}(F) - \{(1/2, \sqrt{3}/2)\}\,, \end{aligned}$$

where F is the circle of radius $\cos(\pi/6)$ and center $(1/2, \sqrt{3}/2)$, and $g_u \circ M_{-\frac{2\pi}{m_u}} \circ g_u^{-1} = M_{-\frac{\pi}{3}}$. So,

$$g_u \circ R_{(u,w)}(\Omega_{uw}) = (\mathrm{int}(C) - \{(0,0)\}) \cap (\mathrm{int}(D) - \{(1,0)\})$$

and

$$g_u \circ R_{(u,w)}(\Omega_{uz}) = (\mathrm{int}(C) - \{(0,0)\}) \cap (\mathrm{int}(F) - \{(1/2, \sqrt{3}/2)\})\,,$$

as shown in Figure A.1.

But, since $M_{-\frac{\pi}{3}}(\mathrm{int}(D) - \{(1,0)\}) = \mathrm{int}(F) - \{(1/2, \sqrt{3}/2)\}$, we get

$$M_{-\pi/3} \circ g_u \circ R_{(u,w)}(\Omega_{uw}) = g_u \circ R_{(u,w)}(\Omega_{uz})\,.$$

To show that $M_{\pi/3} \circ g_u \circ R_{(u,z)}(\Omega_{uz}) = g_u \circ R_{(u,z)}(\Omega_{uw})$, we can proceed as before, but noting that

$$R_{(u,z)} \circ R^{-1}_{(u,w)} = M_{\frac{2\pi}{m_u}} \quad \text{and} \quad g_u \circ M_{\frac{2\pi}{m_u}} \circ g_u^{-1} = M_{\frac{\pi}{3}}\,.$$

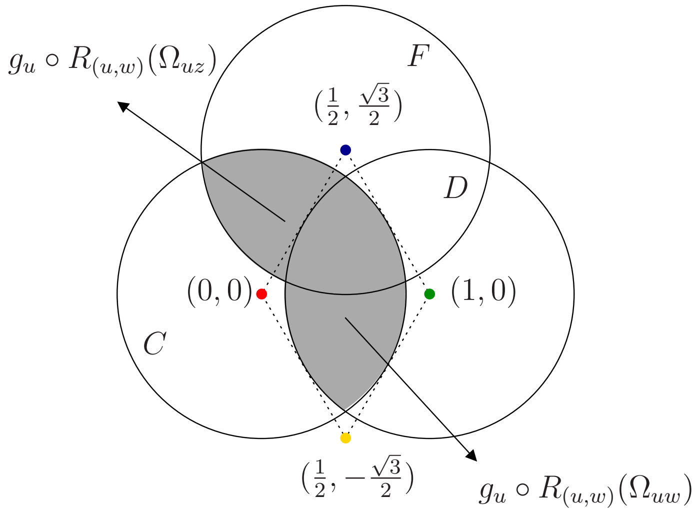

Figure A.1: The sets $g_u \circ R_{(u,w)}(\Omega_{uw})$ and $g_u \circ R_{(u,w)}(\Omega_{uz})$.

To prove the second claim, note that

$$\begin{aligned}
M_{-\frac{2\pi}{m_u}}(\Omega_{uw}) &= M_{-\frac{2\pi}{m_u}}(g_{(w,u)}(\Omega_w - \{(0,0)\}) \cap (\Omega_u - \{(0,0)\})) \\
&= M_{-\frac{2\pi}{m_u}} \circ g_{(w,u)}(\Omega_w - \{(0,0)\}) \cap M_{-\frac{2\pi}{m_u}}(\Omega_u - \{(0,0)\}) \\
&= M_{-\frac{2\pi}{m_u}} \circ R^{-1}_{(u,w)} \circ g_u^{-1} \circ h \circ g_w \circ R_{(w,u)}(\Omega_w - \{(0,0)\}) \cap (\Omega_u - \{(0,0)\}) \\
&= R^{-1}_{(u,z)} \circ g_u^{-1}(\mathrm{int}(D) - \{(0,0)\}) \cap (\Omega_u - \{(0,0)\}) \\
&= \Omega_{uz} \cap (\Omega_u - \{(0,0)\}) \\
&= \Omega_{uz}\,.
\end{aligned}$$

To show that $M_{\frac{2\pi}{m_u}}(\Omega_{uz}) = \Omega_{uw}$ holds, we can proceed as before, but noting that $M_{\frac{2\pi}{m_u}} \circ R^{-1}_{(u,z)} = R^{-1}_{(u,w)}$. □

Lemma 5.8. Let Ω_u, Ω_w, and Ω_x be any three p-domains in $(\Omega_v)_{v \in I}$. If the intersection

$$\Omega_{xu} \cap \Omega_{xw}$$

is nonempty, then

$$\varphi^{-1}_{xu}(\Omega_{xu} \cap \Omega_{xw}) \subseteq \Omega_{uw}\,.$$

Proof. We distinguish three cases: (a) $u = w = x$, (b) $u = w$ and $u \neq x$, or $u = x$ and $u \neq w$, or $w = x$ and $u \neq w$, and (c) $u \neq w$, $u \neq x$, and $w \neq x$. Case (a) is trivial, as $\Omega_{xu} \cap \Omega_{xw} = \Omega_x$, and thus $\varphi^{-1}_{xu}(\Omega_{xu} \cap \Omega_{xw}) = \mathrm{id}_{\Omega_x}(\Omega_x) = \Omega_x = \Omega_{uw} \subseteq \Omega_{uw}$. Case (b) is also trivial. If $u = w$ and $u \neq x$

then $\Omega_{xu} \cap \Omega_{xw} = \Omega_{xu}$, and thus $\varphi_{xu}^{-1}(\Omega_{xu} \cap \Omega_{xw}) = \varphi_{xu}^{-1}(\Omega_{xu}) = \Omega_{ux} \subseteq \Omega_{uw}$. In turn, if $u = x$ and $u \neq w$ then $\Omega_{xu} \cap \Omega_{xw} = \Omega_{xx} \cap \Omega_{xw} = \Omega_{x} \cap \Omega_{xw} = \Omega_{xw}$, and thus $\varphi_{xu}^{-1}(\Omega_{xu} \cap \Omega_{xw}) = \mathrm{id}_{\Omega_x}^{-1}(\Omega_{xw}) = \Omega_{xw} = \Omega_{uw} \subseteq \Omega_{uw}$. Finally, if $w = x$ and $u \neq w$ then $\Omega_{xu} \cap \Omega_{xw} = \Omega_{xu} \cap \Omega_{xx} = \Omega_{xu} \cap \Omega_{x} = \Omega_{xu}$, and thus $\varphi_{xu}^{-1}(\Omega_{xu} \cap \Omega_{xw}) = \varphi_{xu}^{-1}(\Omega_{xu}) = \Omega_{ux} = \Omega_{uw} \subseteq \Omega_{uw}$. So, consider case (c) and assume that the edges $[u, w]$, $[u, x]$, and $[w, x]$ of $\mathcal{K}$ are shared by the triangles $[u, w, x]$ and $[u, w, z]$, $[u, w, x]$ and $[u, x, y]$, and $[u, w, x]$ and $[u, w, v]$ of $\mathcal{K}$, respectively.

The key idea behind our argument is to show that

$$g_{(u,x)}^{-1}(\Omega_{xu} \cap \Omega_{xw}) = \Omega_{ux} \cap \Omega_{uw} \,.$$

In fact, since $g_{(u,x)}^{-1}$ is bijective,

$$g_{(u,x)}^{-1}(\Omega_{xu} \cap \Omega_{xw}) = g_{(u,x)}^{-1}(\Omega_{xu}) \cap g_{(u,x)}^{-1}(\Omega_{xw}) = g_{(x,u)}(\Omega_{xu}) \cap g_{(x,u)}(\Omega_{xw}) = \Omega_{ux} \cap g_{(x,u)}(\Omega_{xw}) \,.$$

By definition,

$$g_{(x,u)}(\Omega_{xw}) = R_{(u,x)}^{-1} \circ g_u^{-1} \circ h \circ g_x \circ R_{(x,u)}(\Omega_{xw}) \,.$$

From Proposition 5.7, we have that

$$R_{(u,x)}^{-1} \circ g_u^{-1} \circ h \circ g_x \circ R_{(x,u)}(\Omega_{xw}) = R_{(u,x)}^{-1} \circ g_u^{-1} \circ h \circ M_{\frac{\pi}{3}} \circ g_x \circ R_{(x,u)}(\Omega_{xu}) \,,$$

where $M_{\frac{\pi}{3}}$ is a rotation by $\frac{\pi}{3}$ around the origin. By construction, the composite function $g_x \circ R_{(x,u)}$ maps Ω_{xu} onto the canonical lens, E, which can be expressed by

$$E = (\mathrm{int}(C) - \{(0,0)\}) \cap (\mathrm{int}(D) - \{(1,0)\}) \,,$$

where C is the circle of radius $\cos(\pi/6)$ and center $(0,0)$ and D is the circle of radius $\cos(\pi/6)$ and center $(1,0)$. So,

$$h \circ M_{\frac{\pi}{3}} \circ g_x \circ R_{(x,u)}(\Omega_{xu})$$

is the set

$$(\mathrm{int}(D) - \{(1,0)\}) \cap \left(\mathrm{int}(G) - \left\{\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)\right\}\right) \,,$$

where G is the circle of radius $\cos(\pi/6)$ and center $(1/2, -\sqrt{3}/2)$. But, only the points of the above set which also belong to $\mathrm{int}(C) - \{(0,0)\}$ are mapped by $R_{(u,x)}^{-1} \circ g_u^{-1}$ to Ω_u. So, we can say that $g_{(x,u)}(\Omega_{xw}) \cap \Omega_u$ is the image of

$$(\mathrm{int}(C) - \{(0,0)\}) \cap (\mathrm{int}(D) - \{(1,0)\}) \cap \left(\mathrm{int}(G) - \left\{\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)\right\}\right)$$

under $R_{(u,x)}^{-1} \circ g_u^{-1}$ (see Figure A.2).

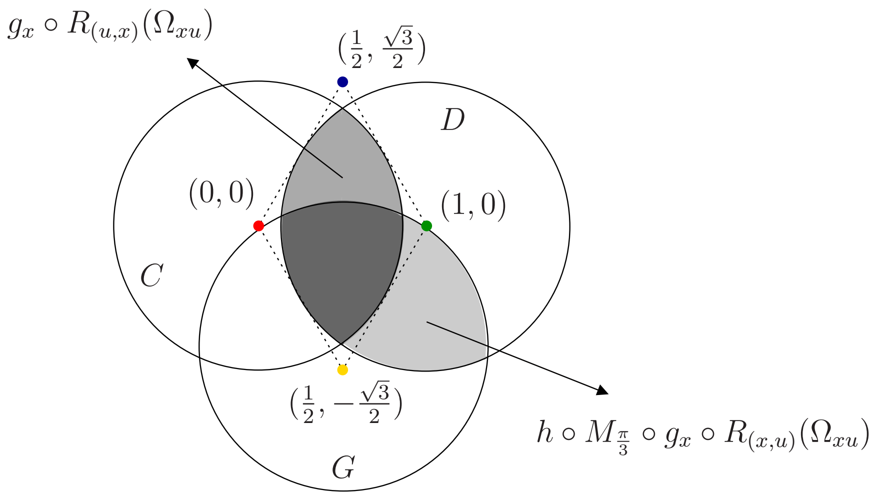

Figure A.2: The sets $h \circ M_{\frac{\pi}{3}} \circ g_x \circ R_{(u,x)} \circ g_u(\Omega_{xu})$ and $h \circ g_x \circ R_{(u,x)} \circ g_u(\Omega_{xu})$.

Now, we claim that the image of $\Omega_{ux} \cap \Omega_{uw}$ under $g_u \circ R_{(u,x)}$ is also equal to

$$(\text{int}(C) - \{(0,0)\}) \cap (\text{int}(D) - \{(1,0)\}) \cap \left(\text{int}(G) - \left\{\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)\right\}\right).$$

In fact,

$$g_u \circ R_{(u,x)}(\Omega_{ux} \cap \Omega_{uw}) = g_u \circ R_{(u,x)}(\Omega_{ux}) \cap g_u \circ R_{(u,x)}(\Omega_{uw}).$$

By definition,

$$g_u \circ R_{(u,x)}(\Omega_{ux}) = E = (\text{int}(C) - \{(0,0)\}) \cap (\text{int}(D) - \{(1,0)\}).$$

In turn, from Proposition 5.7, we know that $g_u \circ R_{(u,x)}(\Omega_{uw}) = M_{-\frac{\pi}{3}} \circ g_u \circ R_{(u,x)}(\Omega_{uw})$. So,

$$g_u \circ R_{(u,x)}(\Omega_{uw}) = M_{-\frac{\pi}{3}}(E) = (\text{int}(C) - \{(0,0)\}) \cap \left(\text{int}(G) - \left\{\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)\right\}\right),$$

and hence

$$g_u \circ R_{(u,x)}(\Omega_{ux} \cap \Omega_{uw}) = (\text{int}(C) - \{(0,0)\}) \cap (\text{int}(D) - \{(1,0)\}) \cap \left(\text{int}(G) - \left\{\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)\right\}\right).$$

This means that

$$\begin{aligned}\Omega_{ux} \cap \Omega_{uw} &= g_{(x,u)}(\Omega_{xw}) \cap \Omega_u \\ &= g_{(x,u)}(\Omega_{xw}) \cap \Omega_{ux} \\ &= g_{(x,u)}(\Omega_{xw}) \cap g_{(x,u)}(\Omega_{xu}) \\ &= g_{(x,u)}(\Omega_{xw} \cap \Omega_{xu}) \\ &= g^{-1}_{(u,x)}(\Omega_{xw} \cap \Omega_{xu}) \,.\end{aligned}$$

Since $\varphi^{-1}_{xu}(p) = g^{-1}_{(u,x)}(p)$, for every $p \in \Omega_{xu}$, we get $\varphi^{-1}_{xu}(\Omega_{xw} \cap \Omega_{xu}) = \Omega_{ux} \cap \Omega_{uw}$, and hence our claim is true. $\square$

Lemma 5.9 Let Ω_u, Ω_w, and Ω_x be any three p-domains in $(\Omega_v)_{v \in I}$. If $\Omega_{xu} \cap \Omega_{xw} \neq \emptyset$, then

$$\varphi_{wu}(p) = \varphi_{wx} \circ \varphi_{xu}(p) \,,$$

for all $p \in \varphi^{-1}_{xu}(\Omega_{xu} \cap \Omega_{xw}) \subseteq \Omega_{uw}$.

Proof. From Lemma 5.8, we know that φ_{wu} is well-defined for all points in $\varphi^{-1}_{xu}(\Omega_{xu} \cap \Omega_{xw}) \subseteq \Omega_{uw}$. So, we are left to show that $\varphi_{wu} = \varphi_{wx} \circ \varphi_{xu}$. We assume that u, w, and x are all distinct; otherwise, if two of them are equal or all of them are the same, our claim would be reduced to condition (3)(b) of Definition 4.1, which has already been proved. Since the indices u, w, and x are assumed to be pairwise distinct, Definition 5.10 tells us that $\varphi_{wu} = g_{(u,w)}$, $\varphi_{wx} = g_{(x,w)}$, and $\varphi_{xu} = g_{(u,x)}$. So, our task amounts to prove that

$$g_{(u,w)}(p) = g_{(x,w)} \circ g_{(u,x)}(p) \,,$$

for all $p \in g^{-1}_{(u,x)}(\Omega_{xu} \cap \Omega_{xw}) \subseteq \Omega_{uw}$.

From Definition 5.8, we know that

$$g_{(u,w)} = R^{-1}_{(w,u)} \circ g_w^{-1} \circ h \circ g_u \circ R_{(u,w)} \,, \tag{A.1}$$

$$g_{(x,w)} = R^{-1}_{(w,x)} \circ g_w^{-1} \circ h \circ g_x \circ R_{(x,w)} \,, \tag{A.2}$$

and

$$g_{(u,x)} = R^{-1}_{(x,u)} \circ g_x^{-1} \circ h \circ g_u \circ R_{(u,x)} \,. \tag{A.3}$$

So,

$$g_{(x,w)} \circ g_{(u,x)} = R^{-1}_{(w,x)} \circ g_w^{-1} \circ h \circ g_x \circ R_{(x,w)} \circ R^{-1}_{(x,u)} \circ g_x^{-1} \circ h \circ g_u \circ R_{(u,x)} \,. \tag{A.4}$$

To show that the right side of Eq. (A.4) is equal to the right side of Eq. (A.1), we make use of Proposition 5.7. So, consider the triangles $[s_u(u), s_u(w), s_u(x)]$, $[s_w(u), s_w(w), s_w(x)]$, and $[s_x(u), s_x(w), s_x(x)]$ of T_u, T_w, and T_x, respectively (see Figure A.3). Without loss of generality, suppose that $s_u(x)$ follows $s_u(w)$ in a counterclockwise traversal of the vertices of P_u. This means that $s_w(u)$ follows $s_w(x)$ in a counterclockwise traversal of the vertices of P_w, and that $s_x(w)$ follows $s_x(u)$ in a counterclockwise traversal of the vertices of P_x.

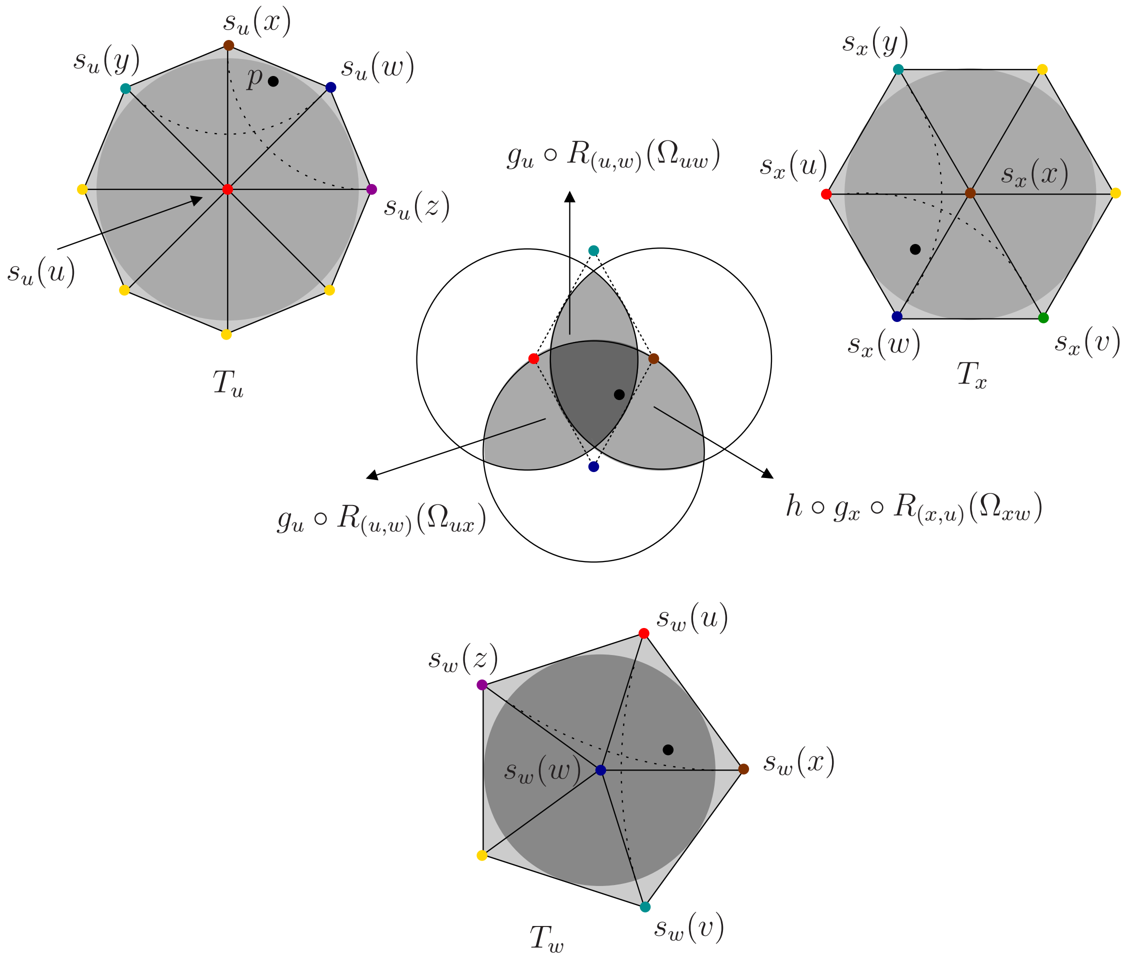

Figure A.3: Illustration of the cocycle condition.

Let p be a point in $g^{-1}_{(u,x)}(\Omega_{xu} \cap \Omega_{xw})$. From Lemma 5.8, we know that $g^{-1}_{(u,x)}(\Omega_{xu} \cap \Omega_{xw}) \subseteq \Omega_{uw}$. From Proposition 5.7, we know that

$$g_u \circ R_{(u,x)}(\Omega_{uw}) = M_{-\frac{\pi}{3}} \circ g_u \circ R_{(u,w)}(\Omega_{uw}) \, ,$$

where $M_{-\frac{\pi}{3}}$ is a rotation by $-\pi/3$ around the origin.

Since $p \in g^{-1}_{(u,x)}(\Omega_{xu} \cap \Omega_{xw})$, we can conclude that

$$g_u \circ R_{(u,x)}(p) = M_{-\frac{\pi}{3}} \circ g_u \circ R_{(u,w)}(p) \, , \tag{A.5}$$

For the same reason, we also know that

$$g_w \circ R_{(w,x)}(q) = M_{\frac{\pi}{3}} \circ g_w \circ R_{(w,u)}(q) \, ,$$

for every $q \in g^{-1}_{(w,x)}(\Omega_{xu} \cap \Omega_{xw}) \subseteq \Omega_{wx}$. So,

$$R^{-1}_{(w,x)} \circ g^{-1}_{w}(t) = R^{-1}_{(w,u)} \circ g^{-1}_{w} \circ M_{-\frac{\pi}{3}}(t)\,, \tag{A.6}$$

for every t such that $t = g_w \circ R_{(w,x)}(q)$, for some $q \in g^{-1}_{(w,x)}(\Omega_{xu} \cap \Omega_{xw})$.

Using the left side of the identities in Eq. (A.5) and Eq. (A.6) to replace their right side in Eq. (A.4), we get

$$g_{(x,w)} \circ g_{(u,x)} = R^{-1}_{(w,u)} \circ g^{-1}_{w} \circ M_{-\frac{\pi}{3}} \circ h \circ g_x \circ R_{(x,w)} \circ R^{-1}_{(x,u)} \circ g^{-1}_{x} \circ h \circ M_{-\frac{\pi}{3}} \circ g_u \circ R_{(u,w)}\,. \tag{A.7}$$

We claim that

$$g_x \circ R_{(x,w)} \circ R^{-1}_{(x,u)} \circ g^{-1}_{x}(q) = M_{-\frac{\pi}{3}}(q)\,,$$

where q is a point in the upper half of the canonical lens, E. To see why, note that

$$R_{(x,w)} \circ R^{-1}_{(x,u)} = M_{-\frac{2\pi}{m_x}}\,,$$

as $s_x(w)$ follows $s_x(u)$ in a counterclockwise traversal of P_x, where m_x is the degree of x. So,

$$g_x \circ R_{(x,w)} \circ R^{-1}_{(x,u)} \circ g^{-1}_{x}(q) = g_x \circ M_{-\frac{2\pi}{m_x}} \circ g^{-1}_{x}(q)\,.$$

But, if (β, s) and (α, t) are the polar coordinates of q and $g_x \circ M_{\frac{2\pi}{m_x}} \circ g^{-1}_{x}(q)$, respectively, then the definition of g_x tells us that

$$\alpha = \frac{m_x}{6} \cdot \left(-\frac{2\pi}{m_x} + \frac{6}{m_x} \cdot \beta \right) = -\frac{\pi}{3} + \beta$$

and

$$t = \frac{\cos(\pi/6)}{\cos(\pi/m_x)} \cdot \frac{\cos(\pi/m_x)}{\cos(\pi/6)} \cdot s = s\,.$$

This implies that

$$g_{(x,w)} \circ g_{(u,x)} = R^{-1}_{(w,u)} \circ g^{-1}_{w} \circ M_{-\frac{\pi}{3}} \circ h \circ M_{-\frac{\pi}{3}} \circ h \circ M_{-\frac{\pi}{3}} \circ g_u \circ R_{(u,w)}\,. \tag{A.8}$$

Finally, we can show that

$$h(p) = M_{-\frac{\pi}{3}} \circ h \circ M_{-\frac{\pi}{3}} \circ h \circ M_{-\frac{\pi}{3}}(p)\,,$$

for every point $p \in \mathbb{R}^2$. This is because

$$h \circ M_{-\frac{\pi}{3}} \circ h \circ M_{-\frac{\pi}{3}} \circ h \circ M_{-\frac{\pi}{3}}$$

is the identity function. But, since $h \circ h$ is the identity function, our claim follows. So,

$$g_{(x,w)} \circ g_{(u,x)}(p) = R^{-1}_{(w,u)} \circ g^{-1}_{w} \circ h \circ g_u \circ R_{(u,w)}(p) = g_{(u,w)}(p)\,, \tag{A.9}$$

for every $p \in g^{-1}_{(u,x)}(\Omega_{xu} \cap \Omega_{xw})$. $\square$

A.2 The Cocycle and Hausdorff Conditions

The cocycle condition (condition 3(c) of Definition 4.1) may seem overly complicated, but it is actually needed to guarantee the transitivity of the relation, $\sim$, defined in the proof of Theorem 4.1. The problem is that $\varphi_{kj} \circ \varphi_{ji}$ is a partial function whose domain, $\varphi_{ji}^{-1}(\Omega_{ji} \cap \Omega_{jk})$, is not necessarily related to the domain, Ω_{ik}, of φ_{ki}. To ensure the transitivity of $\sim$, we must assert that whenever the composition $\varphi_{kj} \circ \varphi_{ji}$ has nonempty domain, this domain is contained in the domain of φ_{ki} and that $\varphi_{kj} \circ \varphi_{ji}$ and φ_{ki} agree.

Flawed versions of condition 3(c) of Definition 4.1 appear in the literature. In particular, Grimm and Hughes [36, 54] uses the following cocycle condition in their definition of a "proto-manifold" (the equivalent of what we call a set of gluing data): *For all $x \in \Omega_{ij} \cap \Omega_{ik}$, we have that $\varphi_{ki}(x) = \varphi_{kj} \circ \varphi_{ji}(x)$.* This condition is not strong enough to imply the transitivity of the relation $\sim$, as shown by the following counterexample:

Consider the open real line intervals $\Omega_1 = (0, 3)$, $\Omega_2 = (4, 5)$, $\Omega_3 = (6, 9)$, $\Omega_{12} = (0, 1)$, $\Omega_{13} = (2, 3)$, $\Omega_{21} = \Omega_{23} = (4, 5)$, $\Omega_{32} = (8, 9)$, and $\Omega_{31} = (6, 7)$, and the transition functions $\varphi_{21}(x) = x+4$, $\varphi_{32}(x) = x+4$, and $\varphi_{31}(x) = x+4$. Note that the pairwise gluings yield Hausdorff spaces. Obviously, we have that $\varphi_{32} \circ \varphi_{21}(x) = x + 8$, for all $x \in \Omega_{12}$, but $\Omega_{12} \cap \Omega_{13} = \emptyset$. Thus, $0.5 \sim 4.5 \sim 8.5$, but $0.5 \not\sim 8.5$ since $\varphi_{31}(0.5)$ is undefined.

A similar and simple example can also be used to show that the Hausdorff condition (condition 4 of Definition 4.1) is necessary. Indeed, let $\Omega_1 = (-3, -1)$, $\Omega_2 = (1, 3)$, $\Omega_{12} = (-3, -2)$, $\Omega_{21} = (1, 2)$, and $\varphi_{21}(x) = x + 4$. The resulting space, M, is a curve looking like a "fork", and the problem is that the images of -2 and 2 in M, which are distinct points of M, cannot be separated. Indeed, the images of any two open intervals, $(-2 - \epsilon, -2 + \epsilon)$ and $(2 - \eta, 2 + \eta)$, for $\epsilon, \eta > 0$, always intersect since $(-2 - \min(\epsilon, \eta), -2)$ and $(2 - \min(\epsilon, \eta), 2)$ are identified. So, M is not Hausdorff. But, as we can clearly see, condition 4 of Definition 4.1 fails.