

## 14.11. Geodesic Lines, Local Gauss-Bonnet Theorem

Geodesics play a very important role in surface theory and in dynamics.

One of the main reasons why geodesics are so important is that they generalize to curved surfaces the notion of “shortest path” between two points in the plane.

**Warning:** as we shall see, this is only true *locally, not globally*.

More precisely, given a surface  $X$ , given any two points  $p = X(u_0, v_0)$  and  $q = X(u_1, v_1)$  on  $X$ , let us look at all the regular curves  $C$  on  $X$  defined on some open interval  $I$  such that  $p = C(t_0)$  and  $q = C(t_1)$  for some  $t_0, t_1 \in I$ .

It can be shown that in order for such a curve  $C$  to minimize the length  $l_C(pq)$  of the curve segment from  $p$  to  $q$ , we must have  $\kappa_g(t) = 0$  along  $[t_0, t_1]$ , where  $\kappa_g(t)$  is the geodesic curvature at  $X(u(t), v(t))$ .

In other words, the principal normal  $\vec{n}$  must be parallel to the normal  $\mathbf{N}$  to the surface along the curve segment from  $p$  to  $q$ .

If  $C$  is parameterized by arc length, this means that the acceleration must be normal to the surface.

It is then natural to define geodesics as those curves such that  $\kappa_g = 0$  everywhere on their domain of definition.

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Actually, there is another way of defining geodesics in terms of vector fields and covariant derivatives (see do Carmo [?] or Berger and Gostiaux [?]), but for simplicity, we stick to the definition in terms of the geodesic curvature.

**Definition 14.11.1** Given a surface  $X: \Omega \rightarrow \mathbb{E}^3$ , a *geodesic line, or geodesic*, is a regular curve  $C: I \rightarrow \mathbb{E}^3$  on  $X$ , such that  $\kappa_g(t) = 0$  for all  $t \in I$ .

Note that by regular curve, we mean that  $\dot{C}(t) \neq 0$  for all  $t \in I$ , i.e.,  $C$  is really a curve, and not a single point.

Physically, a particle constrained to stay on the surface and not acted on by any force, once set in motion with some nonnull initial velocity (tangent to the surface), will follow a geodesic (assuming no friction).

Since  $\kappa_g = 0$  iff the principal normal  $\vec{n}$  to  $C$  at  $t$  is parallel to the normal  $\mathbf{N}$  to the surface at  $X(u(t), v(t))$ , and since the principal normal  $\vec{n}$  is a linear combination of the tangent vector  $\dot{C}(t)$  and the acceleration vector  $\ddot{C}(t)$ , the normal  $\mathbf{N}$  to the surface at  $t$  belongs to the osculating plane.

The differential equations for geodesics are obtained from lemma **14.5.1**.

Since the tangential part of the curvature at a point is given by

$$\kappa_g \vec{n}_g = \left( u_1'' + \sum_{\substack{i=1,2 \\ j=1,2}} \Gamma_{ij}^1 u_i' u_j' \right) X_u + \left( u_2'' + \sum_{\substack{i=1,2 \\ j=1,2}} \Gamma_{ij}^2 u_i' u_j' \right) X_v,$$

the differential equations for geodesics are

$$u_1'' + \sum_{\substack{i=1,2 \\ j=1,2}} \Gamma_{ij}^1 u_i' u_j' = 0,$$
$$u_2'' + \sum_{\substack{i=1,2 \\ j=1,2}} \Gamma_{ij}^2 u_i' u_j' = 0,$$

or more explicitly (letting  $u = u_1$  and  $v = u_2$ ),

$$u'' + \Gamma_{11}^1 (u')^2 + 2\Gamma_{12}^1 u'v' + \Gamma_{22}^1 (v')^2 = 0,$$
$$v'' + \Gamma_{11}^2 (u')^2 + 2\Gamma_{12}^2 u'v' + \Gamma_{22}^2 (v')^2 = 0.$$

In general, it is impossible to find closed-form solutions for these equations.

Nevertheless, from the theory of ordinary differential equations, the following lemma showing the local existence of geodesics can be shown (see do Carmo [?], Chapter 4, Section 4.7):

**Lemma 14.11.2** *Given a surface  $X$ , for every point  $p = X(u, v)$  on  $X$ , for every nonnull tangent vector  $\vec{v} \in T_{(u,v)}(X)$  at  $p$ , there is some  $\epsilon > 0$  and a unique curve  $\gamma: ]-\epsilon, \epsilon[ \rightarrow \mathbb{E}^3$  on the surface  $X$ , such that  $\gamma$  is a geodesic,  $\gamma(0) = p$ , and  $\gamma'(0) = \vec{v}$ .*

To emphasize that the geodesic  $\gamma$  depends on the initial direction  $\vec{v}$ , we often write  $\gamma(t, \vec{v})$  instead of  $\gamma(t)$ .

The geodesics on a sphere are the great circles (the plane sections by planes containing the center of the sphere).

More generally, in the case of a surface of revolution (a surface generated by a plane curve rotating around an axis in the plane containing the curve and not meeting the curve), the differential equations for geodesics can be used to study the geodesics.

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For example, the meridians are geodesics (meridians are the plane sections by planes through the axis of rotation: they are obtained by rotating the original curve generating the surface).

Also, the parallel circles such that at every point  $p$ , the tangent to the meridian through  $p$  is parallel to the axis of rotation, is a geodesic.

In general, there are other geodesics. For more on geodesics on surfaces of revolution, see do Carmo [?], Chapter 4, Section 4, and the problems.

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The geodesics on an ellipsoid are also fascinating, see Berger and Gostiaux [?], Section 10.4.9.5, and Hilbert and Cohn-Vossen [?], Chapter 4, Section 32.

It should be noted that geodesics can be self-intersecting or closed. A deeper study of geodesics requires a study of vector fields on surfaces and would lead us too far.

Technically, what is needed is the exponential map, which we now discuss briefly.

The idea behind the exponential map is to parameterize locally the surface  $X$  in terms of a map from the tangent space to the surface, this map being defined in terms of short geodesics.



More precisely, for every point  $p = X(u, v)$  on the surface, there is some open disk  $B_\epsilon$  of center  $(0, 0)$  in  $\mathbb{R}^2$  (recall that the tangent plane  $T_p(X)$  at  $p$  is isomorphic to  $\mathbb{R}^2$ ), and an injective map

$$\exp_p: B_\epsilon \rightarrow X(\Omega),$$

such that for every  $\vec{v} \in B_\epsilon$  with  $\vec{v} \neq \vec{0}$ ,

$$\exp_p(\vec{v}) = \gamma(1, \vec{v}),$$

where  $\gamma(t, \vec{v})$  is the unique geodesic segment such that  $\gamma(0, \vec{v}) = p$  and  $\gamma'(0, \vec{v}) = \vec{v}$ .

Furthermore, for  $B_\epsilon$  small enough,  $\exp_p$  is a diffeomorphism.

It turns out that  $\exp_p(\vec{v})$  is the point  $q$  obtained by “laying off” a length equal to  $\|\vec{v}\|$  along the unique geodesic that passes through  $p$  in the direction  $\vec{v}$ .

**Lemma 14.11.3** *Given a surface  $X: \Omega \rightarrow \mathbb{E}^3$ , for every  $\vec{v} \neq \vec{0}$  in  $\mathbb{R}^2$ , if*

$$\gamma(-, \vec{v}): ]-\epsilon, \epsilon[ \rightarrow \mathbb{E}^3$$

*is a geodesic on the surface  $X$ , then for every  $\lambda > 0$ , the curve*

$$\gamma(-, \lambda \vec{v}): ]-\epsilon/\lambda, \epsilon/\lambda[ \rightarrow \mathbb{E}^3$$

*is also a geodesic, and*

$$\gamma(t, \lambda \vec{v}) = \gamma(\lambda t, \vec{v}).$$

From lemma 14.11.3, for  $\vec{v} \neq \vec{0}$ , if  $\gamma(1, \vec{v})$  is defined, then

$$\gamma\left(\|\vec{v}\|, \frac{\vec{v}}{\|\vec{v}\|}\right) = \gamma(1, \vec{v}).$$

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This leads to the definition of the exponential map.

**Definition 14.11.4** Given a surface  $X: \Omega \rightarrow \mathbb{E}^3$  and a point  $p = X(u, v)$  on  $X$ , the *exponential map*  $\exp_p$  is the map

$$\exp_p: U \rightarrow X(\Omega)$$

defined such that

$$\exp_p(\vec{v}) = \gamma \left( \|\vec{v}\|, \frac{\vec{v}}{\|\vec{v}\|} \right) = \gamma(1, \vec{v}),$$

where  $\gamma(0, \vec{v}) = p$  and  $U$  is the open subset of  $\mathbb{R}^2 (= T_p(X))$  such that for every  $\vec{v} \neq \vec{0}$ ,  $\gamma \left( \|\vec{v}\|, \frac{\vec{v}}{\|\vec{v}\|} \right)$  is defined. We let  $\exp_p(\vec{0}) = p$ .

It is immediately seen that  $U$  is star-like.

One should realize that in general,  $U$  is a proper subset of  $\Omega$ .

For example, in the case of a sphere, the exponential map is defined everywhere. However, given a point  $p$  on a sphere, if we remove its antipodal point  $-p$ , then  $\exp_p(\vec{v})$  is undefined for points on the circle of radius  $\pi$ .

Nevertheless,  $\exp_p$  is always well-defined in a small open disk.

**Lemma 14.11.5** *Given a surface  $X:\Omega \rightarrow \mathbb{E}^3$ , for every point  $p = X(u, v)$  on  $X$ , there is some  $\epsilon > 0$ , some open disk  $B_\epsilon$  of center  $(0, 0)$ , and some open subset  $V$  of  $X(\Omega)$  with  $p \in V$ , such that the exponential map  $\exp_p: B_\epsilon \rightarrow V$  is well defined and is a diffeomorphism.*

A neighborhood of  $p$  on  $X$  of the form  $\exp_p(B_\epsilon)$  is called a *normal neighborhood* of  $p$ .

The exponential map can be used to define special local coordinate systems on normal neighborhoods, by picking special coordinate systems on the tangent plane.

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In particular, we can use polar coordinates  $(\rho, \theta)$  on  $\mathbb{R}^2$ .

In this case,  $0 < \theta < 2\pi$ . Thus, the closed half-line corresponding to  $\theta = 0$  is omitted, and so is its image under  $\exp_p$ .

It is easily seen that in such a coordinate system,  $E = 1$  and  $F = 0$ , and the  $ds^2$  is of the form

$$ds^2 = dr^2 + G d\theta^2.$$

The image under  $\exp_p$  of a line through the origin in  $\mathbb{R}^2$  is called a *geodesic line*, and the image of a circle centered in the origin is called a *geodesic circle*. Since  $F = 0$ , these lines are orthogonal.

It can also be shown that the Gaussian curvature is expressed as follows:

$$K = -\frac{1}{\sqrt{G}} \frac{\partial^2(\sqrt{G})}{\partial \rho^2}.$$

Polar coordinates can be used to prove the following lemma showing that geodesics locally minimize arc length:



However, globally, geodesics generally do not minimize arc length.

For instance, on a sphere, given any two nonantipodal points  $p, q$ , since there is a unique great circle passing through  $p$  and  $q$ , there are two geodesic arcs joining  $p$  and  $q$ , but only one of them has minimal length.

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**Lemma 14.11.6** *Given a surface  $X: \Omega \rightarrow \mathbb{E}^3$ , for every point  $p = X(u, v)$  on  $X$ , there is some  $\epsilon > 0$  and some open disk  $B_\epsilon$  of center  $(0, 0)$  such that for every  $q \in \exp_p(B_\epsilon)$ , for every geodesic  $\gamma: ] - \eta, \eta[ \rightarrow \mathbb{E}^3$  in  $\exp_p(B_\epsilon)$  such that  $\gamma(0) = p$  and  $\gamma(t_1) = q$ , for every regular curve  $\alpha: [0, t_1] \rightarrow \mathbb{E}^3$  on  $X$  such that  $\alpha(0) = p$  and  $\alpha(t_1) = q$ , then*

$$l_\gamma(pq) \leq l_\alpha(pq),$$

where  $l_\alpha(pq)$  denotes the length of the curve segment  $\alpha$  from  $p$  to  $q$  (and similarly for  $\gamma$ ). Furthermore,  $l_\gamma(pq) = l_\alpha(pq)$  iff the trace of  $\gamma$  is equal to the trace of  $\alpha$  between  $p$  and  $q$ .

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As we already noted, lemma 14.11.6 is false globally, since a geodesic, if extended too much, may not be the shortest path between two points (example of the sphere).

However, the following lemma shows that a shortest path must be a geodesic segment:

**Lemma 14.11.7** *Given a surface  $X: \Omega \rightarrow \mathbb{E}^3$ , let  $\alpha: I \rightarrow \mathbb{E}^3$  be a regular curve on  $X$  parameterized by arc length. For any two points  $p = \alpha(t_0)$  and  $q = \alpha(t_1)$  on  $\alpha$ , assume that the length  $l_\alpha(pq)$  of the curve segment from  $p$  to  $q$  is minimal among all regular curves on  $X$  passing through  $p$  and  $q$ . Then,  $\alpha$  is a geodesic.*

At this point, in order to go further into the theory of surfaces, in particular closed surfaces, it is necessary to introduce differentiable manifolds and more topological tools.

Nevertheless, we can't resist to state one of the “gems” of the differential geometry of surfaces, the local Gauss-Bonnet theorem.

The local Gauss-Bonnet theorem deals with regions on a surface homeomorphic to a closed disk, whose boundary is a closed piecewise regular curve  $\alpha$  without self-intersection.

Such a curve has a finite number of points where the tangent has a discontinuity.

If there are  $n$  such discontinuities  $p_1, \dots, p_n$ , let  $\theta_i$  be the exterior angle between the two tangents at  $p_i$ .

More precisely, if  $\alpha(t_i) = p_i$ , and the two tangents at  $p_i$  are defined by the vectors

$$\lim_{t \rightarrow t_i, t < t_i} \alpha'(t) = \alpha'_-(t_i) \neq \vec{0},$$

and

$$\lim_{t \rightarrow t_i, t > t_i} \alpha'(t) = \alpha'_+(t_i) \neq \vec{0},$$

the angle  $\theta_i$  is defined as follows:

Let  $\theta_i$  be the angle between  $\alpha'_-(t_i)$  and  $\alpha'_+(t_i)$  such that  $0 < |\theta_i| \leq \pi$ , its sign being determined as follows:

If  $p_i$  is not a cusp, which means that  $|\theta_i| \neq \pi$ , we give  $\theta_i$  the sign of the determinant

$$(\alpha'_-(t_i), \alpha'_+(t_i), \mathbf{N}_{p_i}).$$

If  $p_i$  is a cusp, which means that  $|\theta_i| = \pi$ , it is easy to see that there is some  $\epsilon > 0$  such that the determinant

$$(\alpha'(t_i - \eta), \alpha'(t_i + \eta), \mathbf{N}_{p_i})$$

does not change sign for  $\eta \in ] - \epsilon, \epsilon[$ , and we give  $\theta_i$  the sign of this determinant.

Let us call a region defined as above a *simple region*.

In order to state a simpler version of the theorem, let us also assume that the curve segments between consecutive points  $p_i$  are geodesic lines.

We will call such a curve a *geodesic polygon*. Then, the *local Gauss-Bonnet theorem* can be stated as follows:

**Theorem 14.11.8** *Given a surface  $X:\Omega \rightarrow \mathbb{E}^3$ , assuming that  $X$  is injective,  $F = 0$ , and that  $\Omega$  is an open disk, for every simple region  $R$  of  $X(\Omega)$  bounded by a geodesic polygon with  $n$  vertices  $p_1, \dots, p_n$ , letting  $\theta_1, \dots, \theta_n$  be the exterior angles of the geodesic polygon, we have*

$$\iint_R K dA + \sum_{i=1}^n \theta_i = 2\pi.$$

Some clarification regarding the meaning of the integral  $\iint_R K dA$  is in order.

Firstly, it can be shown that the element of area  $dA$  on a surface  $X$  is given by

$$dA = \|X_u \times X_v\| dudv = \sqrt{EG - F^2} dudv.$$

Secondly, if we recall from lemma 14.7.1 that

$$\begin{pmatrix} \mathbf{N}_u \\ \mathbf{N}_v \end{pmatrix} = - \begin{pmatrix} L & M \\ M & N \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} X_u \\ X_v \end{pmatrix},$$

it is easily verified that

$$\mathbf{N}_u \times \mathbf{N}_v = \frac{LN - M^2}{EG - F^2} X_u \times X_v = K(X_u \times X_v).$$

Thus,

$$\begin{aligned} \iint_R K dA &= \iint_R K \|X_u \times X_v\| dudv \\ &= \iint_R \|\mathbf{N}_u \times \mathbf{N}_v\| dudv, \end{aligned}$$

the latter integral representing the area of the spherical image of  $R$  under the Gauss map.

This is the interpretation of the integral  $\iint_R K dA$  that Gauss himself gave.

If the geodesic polygon is a triangle, and if  $A, B, C$  are the interior angles, so that  $A = \pi - \theta_1$ ,  $B = \pi - \theta_2$ ,  $C = \pi - \theta_3$ , the Gauss-Bonnet theorem reduces to what is known as the *Gauss formula*:

$$\iint_R K dA = A + B + C - \pi.$$

The above formula shows that if  $K > 0$  on  $R$ , then  $\iint_R K dA$  is the excess of the sum of the angles of the geodesic triangle over  $\pi$ .

If  $K < 0$  on  $R$ , then  $\iint_R K dA$  is the deficiency of the sum of the angles of the geodesic triangle over  $\pi$ .

And finally, if  $K = 0$ , then  $A + B + C = \pi$ , which we know from the plane!

For the global version of the Gauss-Bonnet theorem, we need the topological notion of the Euler-Poincaré characteristic, but this is beyond the scope of this course.

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## 14.12. Covariant Derivative, Parallel Transport, Geodesics Revisited

Another way to approach geodesics is in terms of covariant derivatives.

The notion of covariant derivative is a key concept of Riemannian geometry, and thus, it is worth discussing anyway.

Let  $X: \Omega \rightarrow \mathbb{E}^3$  be a surface. Given any open subset,  $U$ , of  $X$ , a *vector field on  $U$*  is a function,  $w$ , that assigns to every point,  $p \in U$ , some tangent vector  $w(p) \in T_pX$  to  $X$  at  $p$ .

A vector field,  $w$ , on  $U$  is *differentiable at  $p$*  if, when expressed as  $w = aX_u + bX_v$  in the basis  $(X_u, X_v)$  (of  $T_pX$ ), the functions  $a$  and  $b$  are differentiable at  $p$ .



A vector field,  $w$ , is *differentiable on  $U$*  when it is differentiable at every point  $p \in U$ .

**Definition 14.12.1** Let,  $w$ , be a differentiable vector field on some open subset,  $U$ , of a surface  $X$ . For every  $y \in T_pX$ , consider a curve,  $\alpha: ] - \epsilon, \epsilon[ \rightarrow U$ , on  $X$ , with  $\alpha(0) = p$  and  $\alpha'(0) = y$ , and let  $w(t) = (w \circ \alpha)(t)$  be the restriction of the vector field  $w$  to the curve  $\alpha$ . The normal projection of  $dw/dt(0)$  onto the plane  $T_pX$ , denoted

$$\frac{Dw}{dt}(0), \quad \text{or} \quad D_{\alpha'}w(p), \quad \text{or} \quad D_yw(p),$$

is called the *covariant derivative of  $w$  at  $p$  relative to  $y$* .

The definition of  $Dw/dt(0)$  seems to depend on the curve  $\alpha$ , but in fact, it only depends on  $y$  and the first fundamental form of  $X$ .

Indeed, if  $\alpha(t) = X(u(t), v(t))$ , from

$$w(t) = a(u(t), v(t))X_u + b(u(t), v(t))X_v,$$

we get

$$\frac{dw}{dt} = a(X_{uu}\dot{u} + X_{uv}\dot{v}) + b(X_{vu}\dot{u} + X_{vv}\dot{v}) + \dot{a}X_u + \dot{b}X_v.$$

However, we obtained earlier the following formulae (due to Gauss) for  $X_{uu}$ ,  $X_{uv}$ ,  $X_{vu}$ , and  $X_{vv}$ :

$$\begin{aligned} X_{uu} &= \Gamma_{11}^1 X_u + \Gamma_{11}^2 X_v + LN, \\ X_{uv} &= \Gamma_{12}^1 X_u + \Gamma_{12}^2 X_v + MN, \\ X_{vu} &= \Gamma_{21}^1 X_u + \Gamma_{21}^2 X_v + MN, \\ X_{vv} &= \Gamma_{22}^1 X_u + \Gamma_{22}^2 X_v + NN. \end{aligned}$$

Now,  $Dw/dt$  is the tangential component of  $dw/dt$ , thus, by dropping the normal components, we get

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$$\begin{aligned} \frac{Dw}{dt} &= (\dot{a} + \Gamma_{11}^1 a\dot{u} + \Gamma_{12}^1 a\dot{v} + \Gamma_{21}^1 b\dot{u} + \Gamma_{22}^1 b\dot{v})X_u \\ &\quad + (\dot{b} + \Gamma_{11}^2 a\dot{u} + \Gamma_{12}^2 a\dot{v} + \Gamma_{21}^2 b\dot{u} + \Gamma_{22}^2 b\dot{v})X_v \end{aligned}$$

Thus, the covariant derivative only depends on  $y = (\dot{u}, \dot{v})$ , and the Christoffel symbols, but we know that those only depends on the first fundamental form of  $X$ .

**Definition 14.12.2** Let  $\alpha: I \rightarrow X$  be a regular curve on a surface  $X$ . A *vector field along*  $\alpha$  is a map,  $w$ , that assigns to every  $t \in I$  a vector  $w(t) \in T_{\alpha(t)}X$  in the tangent plane to  $X$  at  $\alpha(t)$ . Such a vector field is differentiable if the components  $a, b$  of  $w = aX_u + bX_v$  over the basis  $(X_u, X_v)$  are differentiable. The expression  $Dw/dt(t)$  defined in the above equation is called the *covariant derivative of  $w$  at  $t$* .

Definition 14.12.2 extends immediately to piecewise regular curves on a surface.

**Definition 14.12.3** Let  $\alpha: I \rightarrow X$  be a regular curve on a surface  $X$ . A vector field along  $\alpha$  is *parallel* if  $Dw/dt = 0$  for all  $t \in I$ .

Thus, a vector field along a curve on a surface is parallel iff its derivative is normal to the surface.

For example, if  $C$  is a great circle on the sphere  $S^2$  parametrized by arc length, the vector field of tangent vectors  $C'(s)$  along  $C$  is a parallel vector field.

We get the following alternate definition of a geodesic.

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**Definition 14.12.4** Let  $\alpha: I \rightarrow X$  be a nonconstant regular curve on a surface  $X$ . Then,  $\alpha$  is a *geodesic* if the field of its tangent vectors,  $\dot{\alpha}(t)$ , is parallel along  $\alpha$ , that is

$$\frac{D\dot{\alpha}}{dt}(t) = 0$$

for all  $t \in I$ .

If we let  $\alpha(t) = X(u(t), v(t))$ , from the equation

$$\begin{aligned} \frac{Dw}{dt} &= (\dot{a} + \Gamma_{11}^1 a\dot{u} + \Gamma_{12}^1 a\dot{v} + \Gamma_{21}^1 b\dot{u} + \Gamma_{22}^1 b\dot{v})X_u \\ &\quad + (\dot{b} + \Gamma_{11}^2 a\dot{u} + \Gamma_{12}^2 a\dot{v} + \Gamma_{21}^2 b\dot{u} + \Gamma_{22}^2 b\dot{v})X_v, \end{aligned}$$

with  $a = \dot{u}$  and  $b = \dot{v}$ , we get the equations

$$\begin{aligned}\ddot{u} + \Gamma_{11}^1(\dot{u})^2 + \Gamma_{12}^1\dot{u}\dot{v} + \Gamma_{21}^1\dot{u}\dot{v} + \Gamma_{22}^1(\dot{v})^2 &= 0 \\ \ddot{v} + \Gamma_{11}^2(\dot{u})^2 + \Gamma_{12}^2\dot{u}\dot{v} + \Gamma_{21}^2\dot{u}\dot{v} + \Gamma_{22}^2(\dot{v})^2 &= 0,\end{aligned}$$

which are indeed the equations of geodesics found earlier, since  $\Gamma_{12}^1 = \Gamma_{21}^1$  and  $\Gamma_{12}^2 = \Gamma_{21}^2$  (except that  $\alpha$  is not necessarily parametrized by arc length).

**Lemma 14.12.5** *Let  $\alpha: I \rightarrow X$  be a regular curve on a surface  $X$ , and let  $v$  and  $w$  be two parallel vector fields along  $\alpha$ . Then, the inner product  $\langle v(t), w(t) \rangle$  is constant along  $\alpha$  (where  $\langle -, - \rangle$  is the inner product associated with the first fundamental form, i.e., the Riemannian metric). In particular,  $\|v\|$  and  $\|w\|$  are constant and the angle between  $v(t)$  and  $w(t)$  is also constant.*

The vector field  $v(t)$  is parallel iff  $dv/dt$  is normal to the tangent plane to the surface  $X$  at  $\alpha(t)$ , and so

$$\langle v'(t), w(t) \rangle = 0$$

for all  $t \in I$ . Similarly, since  $w(t)$  is parallel, we have

$$\langle v(t), w'(t) \rangle = 0$$

for all  $t \in I$ . Then,

$$\langle v(t), w(t) \rangle' = \langle v'(t), w(t) \rangle + \langle v(t), w'(t) \rangle = 0$$

for all  $t \in I$ . which means that  $\langle v(t), w(t) \rangle$  is constant along  $\alpha$ .  $\square$

As a consequence of corollary 14.12.5, if  $\alpha: I \rightarrow X$  is a nonconstant geodesic on  $X$ , then  $\|\dot{\alpha}\| = c$  for some constant  $c > 0$ .

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Thus, we may reparametrize  $\alpha$  w.r.t. the arc length  $s = ct$ , and we note that the parameter  $t$  of a geodesic is proportional to the arc length of  $\alpha$ .

**Lemma 14.12.6** *Let  $\alpha: I \rightarrow X$  be a regular curve on a surface  $X$ , and for any  $t_0 \in I$ , let  $w_0 \in T_{\alpha(t_0)}X$ . Then, there is a unique parallel vector field,  $w(t)$ , along  $\alpha$ , so that  $w(t_0) = w_0$ .*

Lemma 14.12.6 is an immediate consequence of standard results on ODE's. This lemma yields the notion of parallel transport.

**Definition 14.12.7** *Let  $\alpha: I \rightarrow X$  be a regular curve on a surface  $X$ , and for any  $t_0 \in I$ , let  $w_0 \in T_{\alpha(t_0)}X$ . Let  $w$  be the parallel vector field along  $\alpha$ , so that  $w(t_0) = w_0$ , given by Lemma 14.12.6. Then, for any  $t \in I$ , the vector,  $w(t)$ , is called the *parallel transport of  $w_0$  along  $\alpha$  at  $t$* .*

It is easily checked that the parallel transport does not depend on the parametrization of  $\alpha$ . If  $X$  is an open subset of the plane, then the parallel transport of  $w_0$  at  $t$  is indeed a vector  $w(t)$  parallel to  $w_0$  (in fact, equal to  $w_0$ ).

However, on a curved surface, the parallel transport may be somewhat counterintuitive.

If two surfaces  $X$  and  $Y$  are tangent along a curve,  $\alpha: I \rightarrow X$ , and if  $w_0 \in T_{\alpha(t_0)}X = T_{\alpha(t_0)}Y$  is a tangent vector to both  $X$  and  $Y$  at  $t_0$ , then the parallel transport of  $w_0$  along  $\alpha$  is the same, whether it is relative to  $X$  or relative to  $Y$ .

This is because  $Dw/dt$  is the same for both surfaces, and by uniqueness of the parallel transport, the assertion follows.

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This property can be used to figure out the parallel transport of a vector  $w_0$  when  $Y$  is locally isometric to the plane.

In order to generalize the notion of covariant derivative, geodesic, and curvature, to manifolds more general than surfaces, the notion of *connection* is needed.

If  $M$  is a manifold, we can consider the space,  $\mathcal{X}(M)$ , of smooth vector fields,  $X$ , on  $M$ . They are smooth maps that assign to every point  $p \in M$  some vector  $X(p)$  in the tangent space  $T_pM$  to  $M$  at  $p$ .

We can also consider the set  $\mathcal{C}^\infty(M)$  of smooth functions  $f: M \rightarrow \mathbb{R}$  on  $M$ .

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Then, an *affine connection*,  $D$ , on  $M$  is a differentiable map,

$$D: \mathcal{X}(M) \times \mathcal{X}(M) \rightarrow \mathcal{X}(M),$$

denoted  $D_X Y$  (or  $\nabla_X Y$ ), satisfying the following properties:

- (1)  $D_{fX+gY} Z = fD_X Z + gD_Y Z$ ;
- (2)  $D_X(\lambda Y + \mu Z) = \lambda D_X Y + \mu D_X Z$ ;
- (3)  $D_X(fY) = fD_X Y + X(f)Y$ ,

for all  $\lambda, \mu \in \mathbb{R}$ , all  $X, Y, Z \in \mathcal{X}(M)$ , and all  $f, g \in \mathcal{C}^\infty(M)$ , where  $X(f)$  denotes the directional derivative of  $f$  in the direction  $X$ .

Thus, an affine connection is  $\mathcal{C}^\infty(M)$ -linear in  $X$ ,  $\mathbb{R}$ -linear in  $Y$ , and satisfies a “Leibnitz” type of law in  $Y$ .

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For any chart  $\varphi: U \rightarrow \mathbb{R}^m$ , denoting the coordinate functions by  $x_1, \dots, x_m$ , if  $X$  is given locally by

$$X(p) = \sum_{i=1}^m a_i(p) \frac{\partial}{\partial x_i},$$

then

$$X(f)(p) = \sum_{i=1}^m a_i(p) \frac{\partial(f \circ \varphi^{-1})}{\partial x_i}.$$

It can be checked that  $X(f)$  does not depend on the choice of chart.

The intuition behind a connection is that  $D_X Y$  is the directional derivative of  $Y$  in the direction  $X$ .

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The notion of covariant derivative can be introduced via the following lemma:

**Lemma 14.12.8** *Let  $M$  be a smooth manifold and assume that  $D$  is an affine connection on  $M$ . Then, there is a unique map,  $D$ , associating with every vector field  $V$  along a curve  $\alpha: I \rightarrow M$  on  $M$  another vector field,  $DV/dt$ , along  $c$ , so that:*

(1)

$$\frac{D}{dt}(\lambda V + \mu W) = \lambda \frac{DV}{dt} + \mu \frac{DW}{dt}.$$

(2)

$$\frac{D}{dt}(fV) = \frac{df}{dt} V + f \frac{DV}{dt}.$$

(3) *If  $V$  is induced by a vector field  $Y \in \mathcal{X}(M)$ , in the sense that  $V(t) = Y(\alpha(t))$ , then*

$$\frac{DV}{dt} = D_{\alpha'(t)} Y.$$

Then, in local coordinates,  $DV/dt$  can be expressed in terms of the Christoffel symbols, pretty much as in the case of surfaces.

Parallel vector fields, parallel transport, geodesics, are defined as before.

Affine connections are uniquely induced by Riemmanian metrics, a fundamental result of Levi-Civita.

In fact, such connections are *compatible with the metric*, which means that for any smooth curve  $\alpha$  on  $M$  and any two parallel vector fields  $X, Y$  along  $\alpha$ , the inner product  $\langle X, Y \rangle$  is constant.

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Such connections are also *symmetric*, which means that

$$D_X Y - D_Y X = [X, Y],$$

where  $[X, Y]$  is the Lie bracket of vector fields.

For more on all this, consult Do Carmo, or any text on Riemannian geometry.

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