13.11. More on Frenet Frames for $nD$ Curves

Given a curve $f: ]a, b[ \to \mathbb{E}^n$ (or $f: [a, b] \to \mathbb{E}^n$) of class $C^p$, with $p \geq n$, it is interesting to consider families $(e_1(t), \ldots, e_n(t))$ of orthonormal frames. Moreover, if for every $k$, with $1 \leq k \leq n$, the $k$th derivative $f^{(k)}(t)$ of the curve $f(t)$ is a linear combination of $(e_1(t), \ldots, e_k(t))$ for every $t \in ]a, b[,$ then such a frame plays the role of a generalized Frenet frame. This leads to the following definition:

**Definition 13.11.1** Let $f: ]a, b[ \to \mathbb{E}^n$ (or $f: [a, b] \to \mathbb{E}^n$) be a curve of class $C^p$, with $p \geq n$. A family $(e_1(t), \ldots, e_n(t))$ of orthonormal frames, where each $e_i: ]a, b[ \to \mathbb{E}^n$ is $C^{n-i}$-continuous for $i = 1, \ldots, n-1$ and $e_n$ is $C^1$-continuous, is called a *moving frame along* $f$. Furthermore, a moving frame $(e_1(t), \ldots, e_n(t))$ along $f$ so that for every $k$, with $1 \leq k \leq n$, the $k$th derivative $f^{(k)}(t)$ of $f(t)$ is a linear combination of $(e_1(t), \ldots, e_k(t))$ for every $t \in ]a, b[,$ is called a *Frenet n-frame* or *Frenet frame*.
If \((e_1(t), \ldots, e_n(t))\) is a moving frame, then
\[
e_i(t) \cdot e_j(t) = \delta_{ij} \quad \text{for all } i, j, \ 1 \leq i, j \leq n.
\]

**Lemma 13.11.2** Let \(f: ]a, b[ \rightarrow \mathbb{E}^n\) (or \(f: [a, b] \rightarrow \mathbb{E}^n\)) be a curve of class \(C^p\), with \(p \geq n\), so that the derivatives \(f^{(1)}(t), \ldots, f^{(n-1)}(t)\) of \(f(t)\) are linearly independent for all \(t \in ]a, b[\). Then, there is a unique Frenet \(n\)-frame \((e_1(t), \ldots, e_n(t))\) satisfying the following conditions:

1. The \(k\)-frames \((f^{(1)}(t), \ldots, f^{(k)}(t))\) and \((e_1(t), \ldots, e_k(t))\) have the same orientation for all \(k\), with \(1 \leq k \leq n - 1\).

2. The frame \((e_1(t), \ldots, e_n(t))\) has positive orientation.
Proof. Since \( f^{(1)}(t), \ldots, f^{(n-1)}(t) \) is linearly independent, we can use the Gram-Schmidt orthonormalization procedure (see lemma 4.2.7) to construct \( (e_1(t), \ldots, e_{n-1}(t)) \) from \( (f^{(1)}(t), \ldots, f^{(n-1)}(t)) \). We use the generalized cross-product to define \( e_n \), where

\[
e_n = e_1 \times \cdots \times e_{n-1}.
\]

From the Gram-Schmidt procedure, it is easy to check that \( e_k(t) \) is \( C^{n-k} \) for \( 1 \leq k \leq n - 1 \), and since the components of \( e_n \) are certain determinants involving the components of \( (e_1, \ldots, e_{n-1}) \), it is also clear that \( e_n \) is \( C^1 \). □

The Frenet \( n \)-frame given by Lemma 13.11.2 is called the distinguished Frenet \( n \)-frame. We can now prove a generalization of the Frenet-Serret formula that gives an expression of the derivatives of a moving frame in terms of the moving frame itself.
Lemma 13.11.3 Let $f: ]a, b[ \to \mathbb{E}^n$ (or $f: [a, b] \to \mathbb{E}^n$) be a curve of class $C^p$, with $p \geq n$, so that the derivatives $f^{(1)}(t), \ldots, f^{(n-1)}(t)$ of $f(t)$ are linearly independent for all $t \in ]a, b[$. Then, for any moving frame $(e_1(t), \ldots, e_n(t))$, if we write $\omega_{ij}(t) = e'_i(t) \cdot e_j(t)$, we have

$$e'_i(t) = \sum_{j=1}^{n} \omega_{ij}(t)e_j(t),$$

with

$$\omega_{ji}(t) = -\omega_{ij}(t),$$

and there are some functions $\alpha_i(t)$ so that

$$f'(t) = \sum_{i=1}^{n} \alpha_i(t)e_i(t).$$
Furthermore, if \((e_1(t), \ldots, e_n(t))\) is the distinguished Frenet \(n\)-frame associated with \(f\), then we also have

\[
\alpha_1(t) = \|f'(t)\|, \quad \alpha_i(t) = 0 \quad \text{for} \quad i \geq 2,
\]

and

\[
\omega_{ij}(t) = 0 \quad \text{for} \quad j > i + 1.
\]
Proof. Since \((e_1(t), \ldots, e_n(t))\) is a moving frame, it is an orthonormal basis, and thus, \(f'(t)\) and \(e'_i(t)\) are linear combinations of \((e_1(t), \ldots, e_n(t))\). Also, we know that

\[
e'_i(t) = \sum_{j=1}^{n} (e'_i(t) \cdot e_j(t))e_j(t),
\]

and since \(e_i(t) \cdot e_j(t) = \delta_{ij}\), by differentiating, if we write \(\omega_{ij}(t) = e'_i(t) \cdot e_j(t)\), we get

\[
\omega_{ji}(t) = -\omega_{ij}(t).
\]

Now, if \((e_1(t), \ldots, e_n(t))\) is the distinguished Frenet frame, by construction, \(e_i(t)\) is a linear combination of \(f^{(1)}(t), \ldots, f^{(i)}(t)\), and thus \(e'_i(t)\) is a linear combination of \(f^{(2)}(t), \ldots, f^{(i+1)}(t)\), hence of \((e_1(t), \ldots, e_{i+1}(t))\). \(\Box\)
In matrix form, when \((e_1(t), \ldots, e_n(t))\) is the distinguished Frenet frame, the row vector \((e'_1(t), \ldots, e'_n(t))\) can be expressed in terms of the row vector \((e_1(t), \ldots, e_n(t))\) via a skew symmetric matrix \(\omega\), as shown below:

\[
(e'_1(t), \ldots, e'_n(t)) = -(e_1(t), \ldots, e_n(t))\omega(t),
\]

where

\[
\omega = \begin{pmatrix}
0 & \omega_{12} & \omega_{13} & \cdots & \omega_{1n} \\
-\omega_{12} & 0 & \omega_{23} & \cdots & \omega_{2n} \\
-\omega_{13} & -\omega_{23} & 0 & \cdots & \omega_{3n} \\
& \ddots & \ddots & \ddots & \ddots \\
& & & -\omega_{n-1n} & 0 \\
\end{pmatrix}.
\]

The next lemma shows the effect of a reparametrization and of a rigid motion.
Lemma 13.11.4 Let \( f: ]a, b[ \rightarrow \mathbb{E}^n \) (or \( f: [a, b] \rightarrow \mathbb{E}^n \)) be a curve of class \( C^p \), with \( p \geq n \), so that the derivatives \( f^{(1)}(t), \ldots, f^{(n-1)}(t) \) of \( f(t) \) are linearly independent for all \( t \in ]a, b[ \). Let \( h: \mathbb{E}^n \rightarrow \mathbb{E}^n \) be a rigid motion, and assume that the corresponding linear isometry is \( R \). Let \( \tilde{f} = h \circ f \). The following properties hold:

1. For any moving frame \( (e_1(t), \ldots, e_n(t)) \), the \( n \)-tuple \( (\tilde{e}_1(t), \ldots, \tilde{e}_n(t)) \), where \( \tilde{e}_i(t) = R(e_i(t)) \), is a moving frame along \( \tilde{f} \), and we have
   \[ \tilde{\omega}_{ij}(t) = \omega_{ij}(t) \quad \text{and} \quad \|\tilde{f}'(t)\| = \|f'(t)\|. \]

2. For any orientation-preserving diffeomorphism \( \rho: ]c, d[ \rightarrow ]a, b[ \) (i.e., \( \rho'(t) > 0 \) for all \( t \in ]c, d[ \)), if we write \( f = f \circ \rho \), then for any moving frame \( (e_1(t), \ldots, e_n(t)) \) on \( f \), the \( n \)-tuple \( (\tilde{e}_1(t), \ldots, \tilde{e}_n(t)) \), where \( \tilde{e}_i(t) = e_i(\rho(t)) \), is a moving frame on \( f \).
Furthermore, if $\|\tilde{f}'(t)\| \neq 0$, then
\[
\frac{\tilde{\omega}_{ij}(t)}{\|\tilde{f}'(t)\|} = \frac{\omega_{ij}(\rho(t))}{\|f'(\rho(t))\|}.
\]

The proof is straightforward and is omitted.
The above lemma suggests the definition of the curvatures \( \kappa_1, \ldots, \kappa_{n-1} \).

**Definition 13.11.5** Let \( f: ]a, b[ \rightarrow \mathbb{E}^n \) (or \( f: [a, b] \rightarrow \mathbb{E}^n \)) be a curve of class \( C^p \), with \( p \geq n \), so that the derivatives \( f^{(1)}(t), \ldots, f^{(n-1)}(t) \) of \( f(t) \) are linearly independent for all \( t \in ]a, b[ \). If \( (e_1(t), \ldots, e_n(t)) \) is the distinguished Frenet frame associated with \( f \), we define the \textit{i}th curvature, \( \kappa_i \), of \( f \), by

\[
\kappa_i(t) = \frac{\omega_{ii+1}(t)}{\|f'(t)\|},
\]

with \( 1 \leq i \leq n - 1 \).
Observe that the matrix $\omega(t)$ can be written as

$$\omega(t) = \|f'(t)\| \kappa(t),$$

where

$$\kappa = \begin{pmatrix}
0 & \kappa_{12} & & \\
-\kappa_{12} & 0 & \kappa_{23} & \\
& -\kappa_{23} & 0 & \ddots \\
& & \ddots & \kappa_{n-1,n} \\
& & & -\kappa_{n-1,n} & 0
\end{pmatrix}.$$ 

The matrix $\kappa$ is sometimes called the Cartan matrix.

Lemma 13.11.6 Let $f: ]a, b[ \to \mathbb{E}^n$ (or $f: [a, b] \to \mathbb{E}^n$) be a curve of class $C^p$, with $p \geq n$, so that the derivatives $f^{(1)}(t), \ldots, f^{(n-1)}(t)$ of $f(t)$ are linearly independent for all $t \in ]a, b[$. Then for every $i$, with $1 \leq i \leq n-2$, we have $\kappa_i(t) > 0$. 
Proof. Lemma 13.11.2 shows that $e_1, \ldots, e_{n-1}$ are expressed in terms of $f^{(1)}, \ldots, f^{(n-1)}$ by a triangular matrix $(a_{ij})$, whose diagonal entries $a_{ii}$ are strictly positive, i.e., we have

$$e_i = \sum_{j=1}^{i} a_{ij} f^{(j)},$$

for $i = 1, \ldots, n - 1$, and thus,

$$f^{(i)} = \sum_{j=1}^{i} b_{ij} e_j,$$

for $i = 1, \ldots, n - 1$, with $b_{ii} = a_{ii}^{-1} > 0$. Then, since $e_{i+1} \cdot f^{(j)} = 0$ for $j \leq i$, we get

$$\|f'\| \kappa_i = \omega_{i+1} = e_i' \cdot e_{i+1} = a_{ii} f^{(i+1)} \cdot e_{i+1} = a_{ii} b_{i+1} i+1,$$

and since $a_{ii} b_{i+1 i+1} > 0$, we get $\kappa_i > 0 \ (i = 1, \ldots, n - 2)$. □
We conclude by exploring to what extent the curvatures $\kappa_1, \ldots, \kappa_{n-1}$ determine a curve satisfying the nondegeneracy conditions of Lemma 13.11.2. Basically, such curves are defined up to a rigid motion.

**Lemma 13.11.7** Let $f: ]a, b[ \to \mathbb{E}^n$ and $\tilde{f}: ]a, b[ \to \mathbb{E}^n$ (or $f: [a, b] \to \mathbb{E}^n$ and $\tilde{f}: [a, b] \to \mathbb{E}^n$) be two curves of class $C^p$, with $p \geq n$, and satisfying the nondegeneracy conditions of Lemma 13.11.2. Denote the distinguished Frenet frames associated with $f$ and $\tilde{f}$ by $(e_1(t), \ldots, e_n(t))$ and $(\tilde{e}_1(t), \ldots, \tilde{e}_n(t))$. If $\kappa_i(t) = \tilde{\kappa}_i(t)$ for every $i$, with $1 \leq i \leq n - 1$, and $\|f'(t)\| = \|\tilde{f}'(t)\|$ for all $t \in ]a, b[$, then there is a unique rigid motion $h$ so that

$$\tilde{f} = h \circ f.$$
Proof. Fix $t_0 \in ]a,b[$. First of all, there is a unique rigid motion $h$ so that

$$h(f(t_0)) = \tilde{f}(t_0) \quad \text{and} \quad R(e_i(t_0)) = \tilde{e}_i(t_0),$$

for all $i$, with $1 \leq i \leq n$, where $R$ is the linear isometry associated with $h$ (in fact, a rotation). Consider the curve $\bar{f} = h \circ f$. The hypotheses of the lemma and Lemma 13.11.4, imply that

$$\bar{\omega}_{ij}(t) = \tilde{\omega}_{ij}(t) = \omega_{ij}(t), \quad \| \bar{f}'(t) \| = \| \tilde{f}'(t) \| = \| f'(t) \|,$$

and, by construction,

$$(\bar{e}_1(t_0), \ldots, \bar{e}_n(t_0)) = (\tilde{e}_1(t_0), \ldots, \tilde{e}_n(t_0)) \quad \text{and} \quad \bar{f}(t_0) = \tilde{f}(t_0).$$

Let

$$\delta(t) = \sum_{i=1}^{n} (\bar{e}_i(t) - \tilde{e}_i(t)) \cdot (\bar{e}_i(t) - \tilde{e}_i(t)).$$

Then, we have
\[ \delta'(t) = 2 \sum_{i=1}^{n} (e_i(t) - \tilde{e}_i(t)) \cdot (e'_i(t) - \tilde{e}'_i(t)) \]
\[ = -2 \sum_{i=1}^{n} (e_i(t) \cdot \tilde{e}'_i(t) + \tilde{e}_i(t) \cdot e'_i(t)). \]

Using the Frenet equations, we get
\[ \delta'(t) = -2 \sum_{i=1}^{n} \sum_{j=1}^{n} \omega_{ij} e_i \cdot \tilde{e}_j - 2 \sum_{i=1}^{n} \sum_{j=1}^{n} \omega_{ij} \tilde{e}_j \cdot e_i \]
\[ = -2 \sum_{i=1}^{n} \sum_{j=1}^{n} \omega_{ij} e_i \cdot \tilde{e}_j - 2 \sum_{j=1}^{n} \sum_{i=1}^{n} \omega_{ji} e_i \cdot \tilde{e}_j \]
\[ = -2 \sum_{i=1}^{n} \sum_{j=1}^{n} \omega_{ij} e_i \cdot \tilde{e}_j + 2 \sum_{j=1}^{n} \sum_{i=1}^{n} \omega_{ij} e_i \cdot \tilde{e}_j \]
\[ = 0, \]
since \( \omega \) is skew symmetric. Thus, \( \delta(t) \) is constant, and since the Frenet frames at \( t_0 \) agree, we get \( \delta(t) = 0 \).
Then, $\vec{e}_i(t) = \vec{\tilde{e}}_i(t)$ for all $i$, and since $\|\vec{f}'(t)\| = \|\tilde{f}'(t)\|$, we have

$$\vec{f}'(t) = \|\vec{f}'(t)\|\vec{e}_1(t) = \|\tilde{f}'(t)\|\vec{\tilde{e}}_1(t) = \tilde{f}'(t),$$

so that $\vec{f}(t) - \tilde{f}(t)$ is constant. However, $\vec{f}(t_0) = \tilde{f}(t_0)$, and so, $\vec{f}(t) = \tilde{f}(t)$, and $\tilde{f} = \tilde{f} = h \circ f$. □
Lemma 13.11.8 Let $\kappa_1, \ldots, \kappa_{n-1}$ be functions defined on some open $]a, b[\,$ containing 0 with $\kappa_i \, C^{n-i-1}$-continuous for $i = 1, \ldots, n - 1$, and with $\kappa_i(t) > 0$ for $i = 1, \ldots, n - 2$ and all $t \in ]a, b[$. Then, there is curve $f: ]a, b[ \rightarrow \mathbb{E}^n$ of class $C^p$, with $p \geq n$, satisfying the nondegeneracy conditions of Lemma 13.11.2, so that $\|f'(t)\| = 1$ and $f$ has the $n - 1$ curvatures $\kappa_1(t), \ldots, \kappa_{n-1}(t)$.

Proof. Let $X(t)$ be the matrix whose columns are the vectors $e_1(t), \ldots, e_n(t)$ of the Frenet frame along $f$. Consider the system of ODE’s,

$$X'(t) = -X(t)\kappa(t),$$

with initial conditions $X(0) = I$, where $\kappa(t)$ is the skew symmetric matrix of curvatures. By a standard result in ODE’s, there is a unique solution $X(t)$.
We claim that $X(t)$ is an orthogonal matrix. For this, note that

$$
(XX^\top)' = X'X^\top + X(X^\top)' = -X\kappa X^\top - X\kappa^\top X^\top
= -X\kappa X^\top + X\kappa X^\top = 0.
$$

Since $X(0) = I$, we get $XX^\top = I$. If $F(t)$ is the first column of $X(t)$, we define the curve $f$ by

$$
f(s) = \int_0^s F(t) dt,
$$

with $s \in ]a, b[$. It is easily checked that $f$ is a curve parametrized by arc length, with Frenet frame $X(s)$, and with curvatures $\kappa_i$'s. $\square$