14.7. The Gauss Map and its Derivative $d\mathrm{N}$

Given a surface $X\colon \Omega \to \mathbb{E}^3$, given any point $p = X(u, v)$ on X, we have defined the normal $\mathbf{N}_p$ at p (or really $\mathbf{N}_{(u,v)}$ at (u, v)) as the unit vector

$$\mathbf{N}_p = \frac{X_u \times X_v}{\|X_u \times X_v\|}.$$

Gauss realized that the assignment $p \mapsto \mathbf{N}_p$ of the unit normal $\mathbf{N}_p$ to the point p on the surface X could be viewed as a map from the trace of the surface X to the unit sphere S^2.

If $\mathbf{N}_p$ is a unit vector of coordinates (x, y, z), we have $x^2 + y^2 + z^2 = 1$, and $\mathbf{N}_p$ corresponds to the point $N(p) = (x, y, z)$ on the unit sphere.

This is the so-called *Gauss map of* X, denoted as $\mathbf{N}\colon X \to S^2$.

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The derivative $d\mathbf{N}_p$ of the Gauss map at p measures the variation of the normal near p, i.e., how the surface "curves" near p.

The Jacobian matrix of $d\mathbf{N}_p$ in the basis (X_u, X_v) can be expressed simply in terms of the matrices associated with the first and the second fundamental forms (which are quadratic forms).

Furthermore, the eigenvalues of $d\mathbf{N}_p$ are precisely $-\kappa_1$ and $-\kappa_2$, where κ_1 and κ_2 are the principal curvatures at p, and the eigenvectors define the principal directions (when they are well defined).

In view of the negative sign in $-\kappa_1$ and $-\kappa_2$, it is desirable to consider the linear map $\mathcal{S}_p = -d\mathbf{N}_p$, often called the *shape operator*.

Then, it is easily shown that the second fundamental form $\mathrm{II}_p(\overrightarrow{t})$ can be expressed as

$$\mathrm{II}_p(\overrightarrow{t}) = \langle \mathcal{S}_p(\overrightarrow{t}), \overrightarrow{t}\rangle_p,$$

where $\langle -, -\rangle$ is the inner product associated with the first fundamental form.

Thus, the Gaussian curvature is equal to the determinant of $\mathcal{S}_p$, and also to the determinant of $d\mathbf{N}_p$, since $(-\kappa_1)(-\kappa_2) = \kappa_1\kappa_2$.

We will see in a later section that the Gaussian curvature actually only depends of the first fundamental form, which is far from obvious right now!

Actually, if X is not injective, there are problems, because the assignment $p \mapsto \mathbf{N}_p$ could be multivalued.

We can either assume that X is injective, or consider the map from Ω to S^2 defined such that

$$(u, v) \mapsto \mathbf{N}_{(u,v)}.$$

Then, we have a map from Ω to S^2, where (u, v) is mapped to the point $N(u, v)$ on S^2 associated with $\mathbf{N}_{(u,v)}$. This map is denoted as $\mathbf{N}\colon \Omega \to S^2$.

It is interesting to study the derivative $d\mathbf{N}$ of the Gauss map $\mathbf{N}\colon \Omega \to S^2$ (or $\mathbf{N}\colon X \to S^2$).

As we shall see, the second fundamental form can be defined in terms of $d\mathbf{N}$.

For every $(u,v) \in \Omega$, the map $d\mathbf{N}_{(u,v)}$ is a linear map $d\mathbf{N}_{(u,v)}\colon \mathbb{R}^2 \to \mathbb{R}^2$.

It can be viewed as a linear map from the tangent space $T_{(u,v)}(X)$ at $X(u,v)$ (which is isomorphic to $\mathbb{R}^2$) to the tangent space to the sphere at $N(u,v)$ (also isomorphic to $\mathbb{R}^2$).

Recall that $d\mathbf{N}_{(u,v)}$ is defined as follows: For every $(x,y) \in \mathbb{R}^2$,

$$d\mathbf{N}_{(u,v)}(x,y) = \mathbf{N}_u x + \mathbf{N}_v y.$$

Thus, we need to compute $\mathbf{N}_u$ and $\mathbf{N}_v$. Since $\mathbf{N}$ is a unit vector, $\mathbf{N}\cdot\mathbf{N} = 1$, and by taking derivatives, we have $\mathbf{N}_u \cdot \mathbf{N} = 0$ and $\mathbf{N}_v \cdot \mathbf{N} = 0$.

Consequently, $\mathbf{N}_u$ and $\mathbf{N}_v$ are in the tangent space at (u,v), and we can write

$$\begin{aligned} \mathbf{N}_u &= aX_u + cX_v, \\ \mathbf{N}_v &= bX_u + dX_v. \end{aligned}$$

Lemma 14.7.1 *Given a surface X, for any point $p = X(u, v)$ on X, the derivative $d\mathbf{N}_{(u,v)}$ of the Gauss map expressed in the basis (X_u, X_v) is given by the equation*

$$d\mathbf{N}_{(u,v)}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix},$$

where the Jacobian matrix $J(d\mathbf{N}_{(u,v)})$ of $d\mathbf{N}_{(u,v)}$ is given by

$$\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= -\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}\begin{pmatrix} L & M \\ M & N \end{pmatrix} \\ &= \frac{1}{EG - F^2}\begin{pmatrix} MF - LG & NF - MG \\ LF - ME & MF - NE \end{pmatrix}. \end{aligned}$$

The equations

$$\begin{aligned} J(d\mathbf{N}_{(u,v)}) &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= \frac{1}{EG - F^2} \begin{pmatrix} MF - LG & NF - MG \\ LF - ME & MF - NE \end{pmatrix} \end{aligned}$$

are know as the *Weingarten equations* (in matrix form).

If we recall from Section 14.6 the expressions for the Gaussian curvature and for the mean curvature

$$H = \frac{GL - 2FM + EN}{2(EG - F^2)},$$
$$K = \frac{LN - M^2}{EG - F^2},$$

we note that the trace $a + d$ of the Jacobian matrix $J(d\mathbf{N}_{(u,v)})$ of $d\mathbf{N}_{(u,v)}$ is $-2H$, and that its determinant is precisely K.

This is recorded in the following lemma that also shows that the eigenvectors of $J(d\mathbf{N}_{(u,v)})$ correspond to the principal directions:

Lemma 14.7.2 *Given a surface* X*, for any point* $p = X(u,v)$ *on* X*, the eigenvalues of the Jacobian matrix* $J(d\mathbf{N}_{(u,v)})$ *of the derivative* $d\mathbf{N}_{(u,v)}$ *of the Gauss map are* $-\kappa_1, -\kappa_2$*, where* κ_1 *and* κ_2 *are the principal curvatures at* p*, and the eigenvectors of* $d\mathbf{N}_{(u,v)}$ *correspond to the principal directions (when they are defined). The Gaussian curvature* K *is the determinant of the Jacobian matrix of* $d\mathbf{N}_{(u,v)}$*, and the mean curvature* H *is equal to* $-\frac{1}{2}\mathrm{trace}(J(d\mathbf{N}_{(u,v)}))$.

The fact that $\mathbf{N}_u = -\kappa X_u$ when κ is one of the principal curvatures and when X_u corresponds to the corresponding principal direction (and similarly $\mathbf{N}_v = -\kappa X_v$ for the other principal curvature) is known as the formula of Olinde Rodrigues (1815).

The somewhat irritating negative signs arising in the eigenvalues $-\kappa_1$ and $-\kappa_2$ of $d\mathbf{N}_{(u,v)}$ can be eliminated if we consider the linear map $\mathcal{S}_{(u,v)} = -d\mathbf{N}_{(u,v)}$ instead of $d\mathbf{N}_{(u,v)}$.

The map $\mathcal{S}_{(u,v)}$ is called the *shape operator at* p, and the map $d\mathbf{N}_{(u,v)}$ is sometimes called the *Weingarten operator*.

The following lemma shows that the second fundamental form arises from the shape operator, and that the shape operator is self-adjoint with respect to the inner product $\langle -, - \rangle$ associated with the first fundamental form:

Lemma 14.7.3 *Given a surface X, for any point $p = X(u,v)$ on X, the second fundamental form of X at p is given by the formula*

$$II_{(u,v)}(\overrightarrow{t}) = \langle \mathcal{S}_{(u,v)}(\overrightarrow{t}),\ \overrightarrow{t} \rangle,$$

for every $\overrightarrow{t} \in \mathbb{R}^2$. The map $\mathcal{S}_{(u,v)} = -d\mathbf{N}_{(u,v)}$ is self-adjoint, that is,

$$\langle \mathcal{S}_{(u,v)}(\overrightarrow{x}),\ \overrightarrow{y} \rangle = \langle \overrightarrow{x},\ \mathcal{S}_{(u,v)}(\overrightarrow{y}) \rangle,$$

for all $\overrightarrow{x}, \overrightarrow{y} \in \mathbb{R}^2$.

Thus, in some sense, the shape operator contains all the information about curvature.

Remark: The fact that the first fundamental form I is positive definite and that $\mathcal{S}_{(u,v)}$ is self-adjoint with respect to I can be used to give a fancier proof of the fact that $\mathcal{S}_{(u,v)}$ has two real eigenvalues, that the eigenvectors are orthonormal, and that the eigenvalues correspond to the maximum and the minimum of I on the unit circle.

For such a proof, see do Carmo [**?**]. Our proof is more basic and from first principles.

14.8. The Dupin Indicatrix

The second fundamental form shows up again when we study the deviation of a surface from its tangent plane in the neighborhood of the point of tangency.

A way to study this deviation is to imagine that we dip the surface in water, and watch the shorelines formed in the water by the surface in a small region around a chosen point, as we move the surface up and down very gently.

The resulting curve is known as the Dupin indicatrix (1813).

Formally, consider the tangent plane $T_{(u_0,v_0)}(X)$ at some point $p = X(u_0, v_0)$, and consider the perpendicular distance $\rho(u, v)$ from the tangent plane to a point on the surface defined by (u, v).

This perpendicular distance can be expressed as

$$\rho(u, v) = (X(u, v) - X(u_0, v_0)) \cdot \mathbf{N}_{(u_0,v_0)}.$$

However, since X is at least C^3-continuous, by Taylor's formula, in a neighborhood of (u_0, v_0), we can write

$$\begin{aligned} X(u, v) &= X(u_0, v_0) + X_u(u - u_0) + X_v(v - v_0) \\ &\quad + \frac{1}{2}\left(X_{uu}(u - u_0)^2 + 2X_{uv}(u - u_0)(v - v_0) + X_{vv}(v - v_0)^2\right) \\ &\qquad + ((u - u_0)^2 + (v - v_0)^2)h_1(u, v), \end{aligned}$$

where $\lim_{(u,v)\to(u_0,v_0)} h_1(u, v) = 0$.

However, recall that X_u and X_v are really evaluated at (u_0, v_0) (and so are X_{uu}, $X_{u,v}$, and X_{vv}), and so, they are orthogonal to $\mathbf{N}_{(u_0,v_0)}$.

From this, dotting with $\mathbf{N}_{(u_0,v_0)}$, we get

$$\rho(u,v) = \frac{1}{2}\left(L(u-u_0)^2 + 2M(u-u_0)(v-v_0) + N(v-v_0)^2\right) \\ + ((u-u_0)^2 + (v-v_0)^2)h(u,v),$$

where $\lim_{(u,v)\to(u_0,v_0)} h(u,v) = 0$.

Therefore, we get another interpretation of the second fundamental form as a way of measuring the deviation from the tangent plane.

For ϵ small enough, and in a neighborhood of (u_0, v_0) small enough, the set of points $X(u, v)$ on the surface such that $\rho(u, v) = \pm\frac{1}{2}\epsilon^2$ will look like portions of the curves of equation

$$\frac{1}{2}\left(L(u-u_0)^2 + 2M(u-u_0)(v-v_0) + N(v-v_0)^2\right) = \pm\frac{1}{2}\epsilon^2.$$

Letting $u - u_0 = \epsilon x$ and $v - v_0 = \epsilon y$, these curves are defined by the equations

$$Lx^2 + 2Mxy + Ny^2 = \pm 1.$$

These curves are called the *Dupin indicatrix*.

It is more convenient to switch to an orthonormal basis where $\overrightarrow{e_1}$ and $\overrightarrow{e_2}$ are eigenvectors of the Gauss map $d\mathbf{N}_{(u_0,v_0)}$.

If so, it is immediately seen that

$$Lx^2 + 2Mxy + Ny^2 = \kappa_1 x^2 + \kappa_2 y^2,$$

where κ_1 and κ_2 are the principal curvatures. Thus, the equation of the Dupin indicatrix is

$$\kappa_1 x^2 + \kappa_2 y^2 = \pm 1.$$

There are several cases, depending on the sign of $\kappa_1\kappa_2 = K$, i.e., depending on the sign of $LN - M^2$.

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(1) If $LN - M^2 > 0$, then κ_1 and κ_2 have the same sign. This is the case of an *elliptic point*.

If $\kappa_1 \neq \kappa_2$, and $\kappa_1 > 0$ and $\kappa_2 > 0$, we get the ellipse of equation

$$\frac{x^2}{\sqrt{\frac{1}{\kappa_1}}} + \frac{y^2}{\sqrt{\frac{1}{\kappa_2}}} = 1,$$

and if $\kappa_1 < 0$ and $\kappa_2 < 0$, we get the ellipse of equation

$$\frac{x^2}{\sqrt{-\frac{1}{\kappa_1}}} + \frac{y^2}{\sqrt{-\frac{1}{\kappa_2}}} = 1.$$

When $\kappa_1 = \kappa_2$, i.e. an *umbilical point*, the Dupin indicatrix is a circle.

(2) If $LN - M^2 = 0$ and $L^2 + M^2 + N^2 > 0$, then $\kappa_1 = 0$ or $\kappa_2 = 0$, but not both.

This is the case of a *parabolic point*.

In this case, the Dupin indicatrix degenerates to two parallel lines, since the equation is either

$$\kappa_1 x^2 = \pm 1$$

or

$$\kappa_2 y^2 = \pm 1.$$

(3) If $LN - M^2 < 0$ then κ_1 and κ_2 have different signs. This is the case of a *hyperbolic point*.

In this case, the Dupin indicatrix consists of the two hyperbolae of equations

$$\frac{x^2}{\sqrt{\frac{1}{\kappa_1}}} - \frac{y^2}{\sqrt{-\frac{1}{\kappa_2}}} = 1,$$

if $\kappa_1 > 0$ and $\kappa_2 < 0$, or of equation

$$-\frac{x^2}{\sqrt{-\frac{1}{\kappa_1}}} + \frac{y^2}{\sqrt{\frac{1}{\kappa_2}}} = 1,$$

if $\kappa_1 < 0$ and $\kappa_2 > 0$.

These hyperbolae share the same asymptotes, which are the asymptotic directions as defined in Section 14.6, and are given by the equation

$$Lx^2 + 2Mxy + Ny^2 = 0.$$

(4) If $L = M = N$, we have a *planar point*, and in this case, the Dupin indicatrix is undefined.

One should be warned that the Dupin indicatrix for the planar point on the monkey saddle shown in Hilbert and Cohn-Vossen [**?**], Chapter IV, page 192, is wrong!

Therfore, analyzing the shape of the Dupin indicatrix leads us to rediscover the classification of points on a surface in terms of the principal curvatures.

It also lends some intuition to the meaning of the words elliptic, hyperbolic, and parabolic (the last one being a bit misleading).

The analysis of $\rho(u, v)$ also shows that in the elliptic case, in a small neighborhood of $X(u, v)$, all points of X are on the same side of the tangent plane.

This is like being on the top of a round hill.

In the hyperbolic case, in a small neighborhood of $X(u, v)$, there are points of X on both sides of the tangent plane. This is a saddle point, or a valley (or mountain pass).

14.9. The *Theorema Egregium* of Gauss, the Equations of Codazzi-Mainardi, and Bonnet's Theorem

In Section 14.5, we expressed the geodesic curvature in terms of the Christoffel symbols, and we also showed that these symbols only depend on E, F, G, i.e., on the first fundamental form.

In Section 14.7, we expressed $\mathbf{N}_u$ and $\mathbf{N}_v$ in terms of the coefficients of the first and the second fundamental form.

At first glance, given any six functions E, F, G, L, M, N which are at least C^3-continuous on some open subset U of $\mathbb{R}^2$, and where $E, F > 0$ and $EG - F^2 > 0$, it is plausible that there is a surface X defined on some open subset Ω of U, and having
$Ex^2 + 2Fxy + Gy^2$ as its first fundamental form, and
$Lx^2 + 2Mxy + Ny^2$ as its second fundamental form.

However, this is false!

The problem is that for a surface X, the functions E, F, G, L, M, N are not independent.

In this section, we investigate the relations that exist among these functions. We will see that there are three compatibility equations.

The first one gives the Gaussian curvature in terms of the first fundamental form only. This is the famous *Theorema Egregium* of Gauss (1827).

The other two equations express $M_u - L_v$ and $N_u - M_v$ in terms of L, M, N and the Christoffel symbols.

These equations are due to Codazzi (1867) and Mainardi (1856).

Remarkably, these compatibility equations are just what it takes to insure the existence of a surface (at least locally) with $Ex^2 + 2Fxy + Gy^2$ as its first fundamental form, and $Lx^2 + 2Mxy + Ny^2$ as its second fundamental form, an important theorem shown by Ossian Bonnet (1867).

Recall that

$$\begin{aligned} X'' &= X_u u_1'' + X_v u_2'' + X_{uu}(u_1')^2 + 2X_{uv}u_1'u_2' + X_{vv}(u_2')^2, \\ &= (L(u_1')^2 + 2Mu_1'u_2' + N(u_2')^2)\mathbf{N} + \kappa_g \overrightarrow{n_g}, \end{aligned}$$

and since

$$\kappa_g \overrightarrow{n_g} = \left(u_1'' + \sum_{\substack{i=1,2 \\ j=1,2}} \Gamma_{i\,j}^{1}\, u_i' u_j' \right) X_u + \left(u_2'' + \sum_{\substack{i=1,2 \\ j=1,2}} \Gamma_{i\,j}^{2}\, u_i' u_j' \right) X_v,$$

we get the equations (due to Gauss):

$$
\begin{aligned}
X_{uu} &= \Gamma^1_{1\,1} X_u + \Gamma^2_{1\,1} X_v + L\mathbf{N},\\
X_{uv} &= \Gamma^1_{1\,2} X_u + \Gamma^2_{1\,2} X_v + M\mathbf{N},\\
X_{vu} &= \Gamma^1_{2\,1} X_u + \Gamma^2_{2\,1} X_v + M\mathbf{N},\\
X_{vv} &= \Gamma^1_{2\,2} X_u + \Gamma^2_{2\,2} X_v + N\mathbf{N},
\end{aligned}
$$

where the Christoffel symbols $\Gamma^k_{i\,j}$ are defined such that

$$
\begin{pmatrix} \Gamma^1_{i\,j} \\ \Gamma^2_{i\,j} \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} [i\,j;\,1] \\ [i\,j;\,2] \end{pmatrix},
$$

and where

$$
\begin{aligned}
[1\,1;\,1] &= \frac{1}{2}E_u, & [1\,1;\,2] &= F_u - \frac{1}{2}E_v,\\
[1\,2;\,1] &= \frac{1}{2}E_v, & [1\,2;\,2] &= \frac{1}{2}G_u,\\
[2\,1;\,1] &= \frac{1}{2}E_v, & [2\,1;\,2] &= \frac{1}{2}G_u,\\
[2\,2;\,1] &= F_v - \frac{1}{2}G_u, & [2\,2;\,2] &= \frac{1}{2}G_v.
\end{aligned}
$$

Also, recall from Section 14.7 that we have the Weingarten equations

$$\begin{aligned}\begin{pmatrix}\mathbf{N}_u\\ \mathbf{N}_v\end{pmatrix} &= \begin{pmatrix}a & c\\ b & d\end{pmatrix}\begin{pmatrix}X_u\\ X_v\end{pmatrix}\\ &= -\begin{pmatrix}L & M\\ M & N\end{pmatrix}\begin{pmatrix}E & F\\ F & G\end{pmatrix}^{-1}\begin{pmatrix}X_u\\ X_v\end{pmatrix}.\end{aligned}$$

From the Gauss equations and the Weingarten equations

$$\begin{aligned}X_{uu} &= \Gamma_{1\,1}^{1}X_u + \Gamma_{1\,1}^{2}X_v + L\mathbf{N},\\ X_{uv} &= \Gamma_{1\,2}^{1}X_u + \Gamma_{1\,2}^{2}X_v + M\mathbf{N},\\ X_{vu} &= \Gamma_{2\,1}^{1}X_u + \Gamma_{2\,1}^{2}X_v + M\mathbf{N},\\ X_{vv} &= \Gamma_{2\,2}^{1}X_u + \Gamma_{2\,2}^{2}X_v + N\mathbf{N},\\ \mathbf{N}_u &= aX_u + cX_v,\\ \mathbf{N}_v &= bX_u + dX_v,\end{aligned}$$

we see that the partial derivatives of X_u, X_v and $\mathbf{N}$ can be expressed in terms of the coefficient E, F, G, L, M, N and their partial derivatives.

Thus, a way to obtain relations among these coefficients is to write the equations expressing the commutation of partials, i.e.,

$$\begin{aligned}(X_{uu})_v - (X_{uv})_u &= 0,\\ (X_{vv})_u - (X_{vu})_v &= 0,\\ \mathbf{N}_{uv} - \mathbf{N}_{vu} &= 0.\end{aligned}$$

Using the Gauss equations and the Weingarten equations, we obain relations of the form

$$\begin{aligned}A_1 X_u + B_1 X_v + C_1 \mathbf{N} &= 0,\\ A_2 X_u + B_2 X_v + C_2 \mathbf{N} &= 0,\\ A_3 X_u + B_3 X_v + C_3 \mathbf{N} &= 0,\end{aligned}$$

where A_i, B_i, and C_i are functions of E, F, G, L, M, N and their partial derivatives, for $i = 1, 2, 3$.

However, since the vectors X_u, X_v, and $\mathbf{N}$ are linearly independent, we obtain the nine equations

$$A_i = 0, \quad B_i = 0, \quad C_i = 0, \quad \text{for } i = 1, 2, 3.$$

Although this is very tedious, it can be shown that these equations are equivalent to just three equations.

Due to its importance, we state the *Theorema Egregium* of Gauss.

Theorem 14.9.1 *Given a surface X and a point $p = X(u, v)$ on X, the Gaussian curvature K at (u, v) can be expressed as a function of E, F, G and their partial derivatives. In fact*

$$(EG - F^2)^2 K =$$

$$\begin{vmatrix} C & F_v - \frac{1}{2}G_u & \frac{1}{2}G_v \\ \frac{1}{2}E_u & E & F \\ F_u - \frac{1}{2}E_v & F & G \end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2}E_v & \frac{1}{2}G_u \\ \frac{1}{2}E_v & E & F \\ \frac{1}{2}G_u & F & G \end{vmatrix}$$

where

$$C = \frac{1}{2}(-E_{vv} + 2F_{uv} - G_{uu}).$$

Proof. Following Darboux [?] (Volume III, page 246), a way of proving theorem 14.9.1 is to start from the formula

$$K = \frac{LN - M^2}{EG - F^2}$$

and to go back to the expressions of L, M, N using D, D', D'' as determinants:

$$L = \frac{D}{\sqrt{EG - F^2}}, \quad M = \frac{D'}{\sqrt{EG - F^2}}, \quad N = \frac{D''}{\sqrt{EG - F^2}},$$

where

$$\begin{aligned} D &= (X_u, X_v, X_{uu}), \\ D' &= (X_u, X_v, X_{uv}), \\ D'' &= (X_u, X_v, X_{vv}). \end{aligned}$$

Then, we can write

$$(EG - F^2)^2 K = (X_u, X_v, X_{uu})(X_u, X_v, X_{vv}) - (X_u, X_v, X_{uv})^2,$$

and compute these determinants by multiplying them out. One will eventually get the expression given in the theorem! ☐

It can be shown that the other two equations, known as the *Codazzi-Mainardi equations*, are the equations:

$$\begin{aligned} M_u - L_v &= \Gamma^2_{1\,1}N - (\Gamma^2_{1\,2} - \Gamma^1_{1\,1})M - \Gamma^1_{1\,2}L, \\ N_u - M_v &= \Gamma^2_{1\,2}N - (\Gamma^2_{2\,2} - \Gamma^1_{1\,2})M - \Gamma^1_{2\,2}L. \end{aligned}$$

We conclude this section with an important theorem of Ossian Bonnet. First, we show that the first and the second fundamental forms determine a surface up to rigid motion. More precisely, we have the following lemma:

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Lemma 14.9.2 *Let* $X\colon\Omega \to \mathbb{E}^3$ *and* $Y\colon\Omega \to \mathbb{E}^3$ *be two surfaces over a connected open set* Ω. *If* X *and* Y *have the same coefficients* E, F, G, L, M, N *over* Ω, *then there is a rigid motion mapping* $X(\Omega)$ *onto* $Y(\Omega)$.

The above lemma can be shown using a standard theorem about ordinary differential equations (see do Carmo, [**?**] Appendix to Chapter 4, pp. 309-314). Finally, we state Bonnet's theorem.

Theorem 14.9.3 *Let* $E, F, G,\ L, M, N$ *be any* C^3*-continuous functions on some open set* $U \subset \mathbb{R}^2$*, and such that* $E > 0$*,* $G > 0$*, and* $EG - F^2 > 0$*. If these functions satisfy the Gauss formula (of the Theorema Egregium) and the Codazzi-Mainardi equations, then for every* $(u, v) \in U$*, there is an open set* $\Omega \subseteq U$ *such that* $(u, v) \in \Omega$*, and a surface* $X\colon \Omega \to \mathbb{E}^3$ *such that* X *is a diffeomorphism, and* E, F, G *are the coefficients of the first fundamental form of* X*, and* L, M, N *are the coefficients of the second fundamental form of* X*. Furthermore, if* Ω *is connected, then* $X(\Omega)$ *is unique up to a rigid motion.*

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