1. (a) We construct the desired ϵ-NFA in pieces as follows:

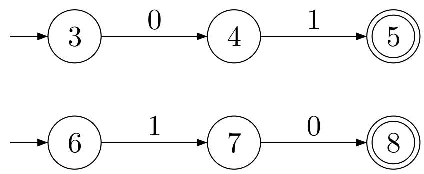

Figure 1: DFAs that recognizes 01 and 10.

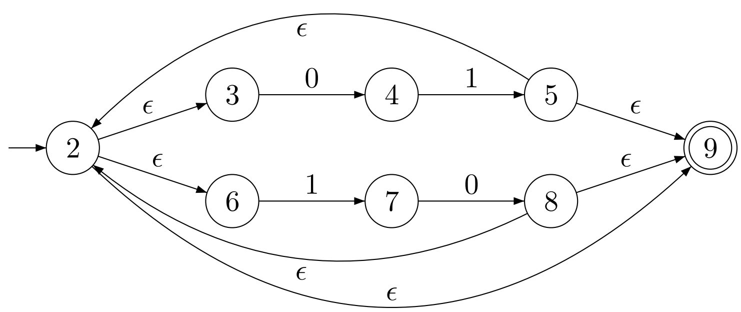

Figure 2: DFA that recognizes $(01 + 10)^*$.

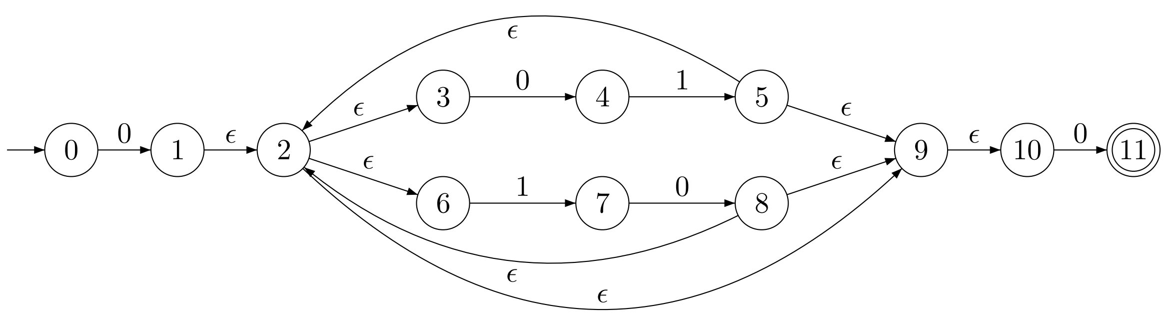

Figure 3: DFA that recognizes $0(01 + 10)^*0$.

(b) With the above ϵ-NFA, we trace through its execution noting the set of states that we're in and add to our DFA accordingly until there are no more states with inputs left to consider. This results in the given DFA.

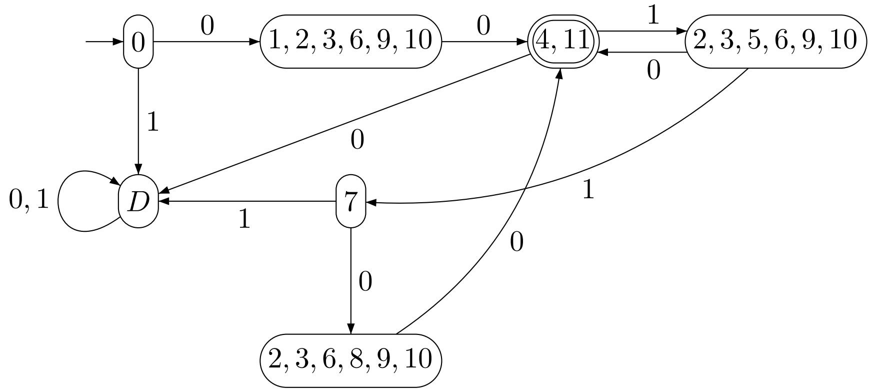

Figure 4: The DFA translation of our ϵ-NFA above.

2. We proceed via the state elimination algorithm outlined by Sipser:

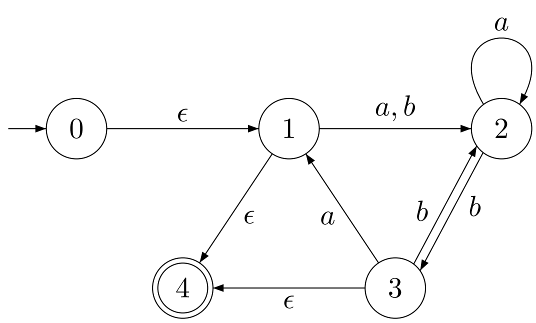

Figure 5: Add a unique start state and accept state to the NFA.

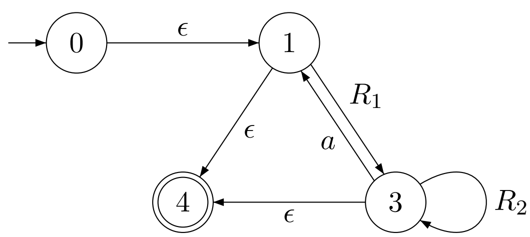

Figure 6: Eliminate state 2 with $R_1 = (a+b)a^*b$ and $R_2 = ba^*b$.

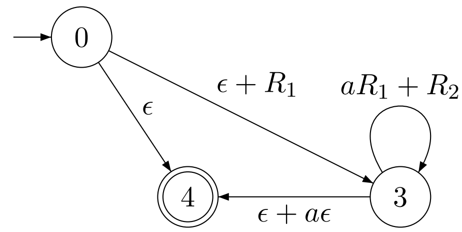

Figure 7: Eliminate state 1.

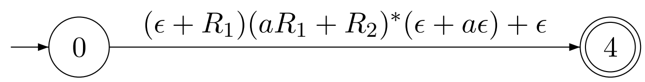

Figure 8: Eliminate state 3.

From the final diagram, we can conclude that our regular expression is $R_1(aR_1+R_2)^*(\epsilon+a)+\epsilon$ with $R_1 = (a+b)a^*b$ and $R_2 = ba^*b$.

3. (a) We prove that $L = \{0^m1^n \mid m < n\}$ is not regular via the pumping lemma. Let p be the pumping length and $w = 0^p1^{p+1}$ which satisfies the pumping lemma since $|w| > p$. Consider decompositions of $w = xyz$ such that $|xy| \le p$ and $|y| \ge 1$. From the shape of w, we can conclude that xy must reside in the 0^p portion of w and thus y consists of at least one 0 and only 0s.

   Consider $w' = xyyz$ which the pumping lemma says is in L. Since y contains at least one 0 then w' must have at least $p+1$ 0s. But w' has $p+1$ 1s so the number of 0s is at least equal to the number of 1s. Therefore $w' \notin L$, a contradiction.

   (b) We prove that $L = \{0^n \mid n \text{ is a power of } 2\}$ is not regular via the pumping lemma. Let p be the pumping length and $w = 0^{2^p}$ which satisfies the pumping lemma since $|w| > p$. Consider decompositions of $w = xyz$ such that $|xy| \le p$ and $|y| \ge 1$. Since y is non-empty it must have the shape 0^m for $1 \le m \le p$.

   Consider $w' = xyyz$ which the pumping lemma says is in L. $|w'| \le 2^p + p$ since $w' = 0^{2^p+m}$ where m is bounded above by p. Note that 2^{p+1} is the next power of 2 after 2^p so $w'' = 0^{2^{p+1}} \in L$. It follows that $|w'| < |w''|$ since $2^p + p < 2^{p+1}$ for $p > 0$. Furthermore it is clear that $|w| < |w'|$ because y must be non-empty.

   Therefore, we can conclude that $|w| < |w'| < |w''|$. However we asserted that $|w''|$ is the next power of 2 after $|w|$ and thus $|w'|$ cannot be a power of 2 and consequently $w' \notin L$, a contradiction.

4. To prove that $\mathsf{alt}(L, M) = \{a_1 \cdots a_n \mid n \text{ is even}, a_1a_3\cdots a_{n-1} \in L, a_2a_4\cdots a_n \in M\}$ is regular we construct a DFA D that recognizes $\mathsf{alt}(L, M)$. Our construction is similar to a pairs construction of two DFAs except we alternate simulating the DFAs for L and M instead of simulating them in parallel. To decide which machine should run next in our single-step function δ, we encode the parity of the string seen so far in our states.

Let $D_1 = (Q^1, \delta^1, q_0^1, F^1)$ recognize L and $D_2 = (Q^2, \delta^2, q_0^2, F^2)$ recognize M. Construct $D = (Q, \delta, q_0, F)$ that recognizes $\mathsf{alt}(L, M)$ as follows:

- $Q = Q^1 \times Q^2 \times \{\mathsf{even}, \mathsf{odd}\}$. A state of Q encapsulates the current state in D_1, the current state in D_2, and the parity (denoted p with values even, odd) of the word seen so far. For notational convenience let the triple $(q_i^1, q_j^2, p) \equiv q_{i,j}^p$.
- Define δ as

$$\delta(q_{i,j}^p, x) = \begin{cases} q_{k,j}^{\mathsf{odd}} & \text{where } \delta^1(q_i^1, x) = q_k^1 \text{ if } p = \mathsf{even} \\ q_{i,k}^{\mathsf{even}} & \text{where } \delta^2(q_j^2, x) = q_k^2 \text{ if } p = \mathsf{odd}. \end{cases}$$

  The transition function can be read as "If we've seen an $\{\mathsf{even}, \mathsf{odd}\}$ number of characters then x is {odd, even} and thus run it on $\{D_1, D_2\}$".

- $q_0 = q_{0,0}^{\mathsf{even}}$.
- $F = \{q_{i,j}^{\mathsf{even}} \mid q_i^1 \in F^1, q_j^2 \in F_2\}$. The set of accepting states are those in which the last two characters, i.e., the last characters read by D_1 and D_2, lead to accepting states and the overall word is even.

The level of explanation above is sufficient for the problem. To rigorously prove that D accepts $\mathsf{alt}(L, M)$ we must prove a lemma similar to the following

Lemma. If $\hat{\delta}(q_{i,j}^p, a_1 \cdots a_n) = q_{k,l}^{p'}$ then $\delta^1(q_i^1, x) = q_k^1$ and $\delta^2(q_j^2, y) = q_l^2$ where

$$(x, y, p') = \begin{cases} (a_1 a_3 \cdots a_{n-1}, a_2 a_4 \cdots a_n, \mathsf{even}) & \text{if } n \text{ is even} \\ (a_1 a_2 \cdots a_n, a_2 a_4 \cdots a_{n-1}, \mathsf{odd}) & \text{if } n \text{ is odd} \end{cases}$$

which formalizes our description of the meaning of Q and δ. The case analysis in the lemma is important as we must consider all possible input strings to D in our induction, not just those with even lengths.

From this lemma and our definition of F we can conclude that D accepts $\mathsf{alt}(L, M)$.