CIS 262 Fall 2016: Problem Set 2, Due 11:59 AM September 27

Please write your answers succinctly and rigorously. Remember that all submissions must go through GradeScope. Problem 10 will be done on Automatatutor using the same account you made for problemset 1. You do not need to turn anything in on GradeScope for Problem 10, since it will be graded by AutomataTutor.

1. Let \( \Sigma = \{a, b\} \). Consider the language \( L \) consisting of strings \( w \) such that \( w \) has at least one occurrence of the substring \( aba \).
   (a) Give a DFA that accepts \( L \). 4 pts
   (b) Prove that the DFA of part (a) accepts \( L \). That is, show that for every string \( w \), \( \hat{\delta}(q_0, w) \) is an accepting state precisely when \( w \) has at least one occurrence of the substring \( aba \). 6 pts

2. (Sipser 1.14)
   (a) Show that if \( M \) is a DFA recognizing language \( B \), swapping the accept and nonaccept states in \( M \) yields a new DFA \( \tilde{M} \) recognizing \( \tilde{B} \), the complement of \( B \). Conclude that the class of regular languages is closed under complement. 5 pts
   (b) Show by example that if \( M \) is an NFA recognizing language \( B \), \( \tilde{M} \) does not necessarily recognize \( \tilde{B} \). 5 pts

3. (Sipser 1.31) For any string \( w = w_1w_2 \cdots w_n \), the reverse of \( w \) is \( w^R = w_n \cdots w_2w_1 \). For any language \( L \), let \( L^R = \{w^R \mid w \in L\} \). Show that \( L \) is regular \( \iff \) \( L^R \) is regular. 10 pts

4. (Sipser 1.36) Let \( B_n = \{a^k \mid k \text{ is a multiple of } n\} \). Show that, for each natural number \( n \geq 1 \), the language \( B_n \) is regular. 5 pts

5. (Sipser 1.48) Given alphabet \( \Sigma = \{a, b\} \), prove that
   \[
   L = \{w \mid w \text{ contains an equal number of occurrences of the substrings } ab \text{ and } ba\}
   \]
is regular. 5 pts

6. (Sipser 1.51) Let \( x \) and \( y \) be strings in some language \( L \). As discussed in class, we say that \( x \) and \( y \) are distinguishable by \( L \) if some string \( z \) exists such that exactly one of strings \( xz \) and \( yz \) is a member of \( L \). Otherwise, for every string \( z \) we have \( xz \in L \) whenever \( yz \in L \) and we say that \( x \) and \( y \) are indistinguishable by \( L \), written \( x \equiv_L y \).
   Prove that \( \equiv_L \) is an equivalence relation. 10 pts

7. Since we proved that \( \equiv_L \) is an equivalence relation in Question 6, it is natural to consider the equivalence classes induced by \( \equiv_L \). Recalling that given language \( L \) and string \( x \), the equivalence class to which \( x \) belongs is the set of strings \( \{y \mid x \equiv_L y\} \). \( \equiv_L \) therefore partitions \( L \) into some number of equivalence classes.
   Prove that if \( L \) has an infinite number of equivalence classes under \( \equiv_L \) then it cannot be recognized by a DFA. 10 pts
8. Given alphabet $\Sigma = \{a, b\}$, show that any DFA accepting the language $L = \{w \in \Sigma^* \mid \text{count}(w, a) \equiv 2 \text{ or } 4 \mod 5\}$ must have at least two final states.

9. Given alphabet $\Sigma = \{a, b\}$ and language $L = \{w \mid \text{count}(w, a) \geq k\}$ for some natural number $k$, what is the minimum number of states of a DFA $M$ accepting $L$? Prove your answer.

10. Log into Automata Tutor and complete the NFA construction problems.