7-3. 
\[ E[X - Y|\alpha] = \int_0^1 \int_0^x |x - y|^{\alpha} \, dx \, dy. \] 
Now 
\[ \int_0^1 |x - y|^{\alpha} \, dy = \int_0^x (x - y)^{\alpha} \, dy + \int_0^1 (y - x)^{\alpha} \, dy \]
\[ = \int_0^x u^{\alpha} \, du + \int_0^{1-x} u^{\alpha} \, du \]
\[ = [x^{\alpha+1} + (1-x)^{\alpha+1}] / (\alpha + 1) \]

Hence,
\[ E[X - Y|\alpha] = \frac{1}{\alpha + 1} \int_0^1 [x^{\alpha+1} + (1-x)^{\alpha+1}] \, dx = \frac{2}{(\alpha + 1)(\alpha + 2)} \]

7-5 The joint density of the point \((X,Y)\) at which the accident occurs is
\[ f(x, y) = \frac{1}{9}, \quad -3/2 < x, y < 3/2 \]
\[ = f(x)f(y) \]

Where \(f(a) = \frac{1}{3}, \quad -3/2 < a << 3/2\)

Hence we may conclude that \(X\) and \(Y\) are independent and uniformly distributed on the interval \((-3/2, 3/2)\). Therefore
\[ E[|X|] + E[|Y|] = 2 \int_{-3/2}^{3/2} \frac{1}{3} x \, dx + 4 \int_0^{3/2} x \, dx = \frac{3}{2} \]

7-8.
\[ E[\text{number of occupied tables}] = E\left[ \sum_{i=1}^N X_i \right] = \sum_{i=1}^N E[X_i] \]

Now,
\[ E[X_i] = P\{i^{th} \text{ arrival is not friends with any of first } i-1\} = (1 - p)^{i-1} \]

and so
\[ E[\text{number of occupied tables}] = \sum_{i=1}^N (1 - p)^{i-1} \]
7-13 Let $X_i$ be the indicator for the event that person $I$ is given a card whose number matches his age. Because only one of the cards matches the age of the person $I$

$$E\left[\sum_{i=1}^{1000} X_i\right] = \sum_{i=1}^{1000} E[X_i] = 1$$

7-14 The number of stages is a negative binomial random variable with parameters $m$ and $1-p$. Hence, its expected value is $m(1-p)$

7-18 $E$[number of matches]$= E\left[\sum_{i=1}^{52} I_i\right]$, $I_i = \begin{cases} 1 & \text{match on card } i \\ 0 & \text{no match on card } i \end{cases}$

$$= 52 \cdot \frac{1}{13} = 4, \text{ since } E[I_i] = 1/13$$

7-23

$$E\left[\sum_{i=1}^{5} X_i + \sum_{i=1}^{8} Y_i\right] = \sum_{i=1}^{5} E[X_i] + \sum_{i=1}^{8} E[Y_i] = 5 \cdot \frac{3}{11} + 3 \cdot \frac{3}{120} = \frac{147}{110}$$

7-27 Let $X$ denote the number of items in a randomly chosen box. Then, with $X_i$ equal to 1 if item $I$ is in the randomly chosen box

$$E[X] = E\left[\sum_{i=1}^{101} X_i\right] = \sum_{i=1}^{101} E[X_i] = \frac{101}{10} > 10$$

Hence, $X$ can exceed 10, showing that at least one of the boxed must contain more than 10 items.

7-39.

Given that $Y_n = X_n + X_{n+1} + X_{n+2}$ and $Y_{n+1} = X_{n+1} + X_{n+2} + X_{n+3}$

Let $Y_n = X_n + X_{n+1} + X_{n+2} = X_n + Z$ and $Y_{n+1} = Z + X_{n+3}$

Then from Proposition 3.2 in Chapter 7 of the text, the following is fairly obvious

$$Cov(Y_n, Y_n) = Var(Y_n) = 3\sigma^2$$

$$Cov(Y_n, Y_{n+1}) = Cov(X_n + X_{n+1} + X_{n+2}, X_{n+1} + X_{n+2} + X_{n+3}) = Cov(X_{n+1} + X_{n+2}, X_{n+1} + X_{n+2}) = Var(X_{n+1} + X_{n+2}) = 2\sigma^2$$

$$Cov(Y_n, Y_{n+2}) = Cov(X_{n+2}, X_{n+2}) = \sigma^2$$

$$Cov(Y_n, Y_{n+j}) = 0 \text{ when } j \geq 3$$

7-42
a) Let \( X_i = \begin{cases} 1 & \text{pair } i \text{ consists of a man and a woman} \\ 0 & \text{otherwise} \end{cases} \)

\[
E[X_i] = P[X_i = 1] = \frac{10}{19}
\]

\[
E[X_i X_j] = P[X_i = 1, X_j = 1] = P(X_i = 1)P(X_j = 1|X_2 = 1) = \frac{10 \cdot 9}{19 \cdot 17}, \quad i \neq j
\]

\[
E\left[ \sum_{i=1}^{10} X_i \right] = \frac{100}{19}
\]

\[
Var\left[ \sum_{i=1}^{10} X_i \right] = 10 \cdot \frac{10}{19} \left( 1 - \frac{10}{19} \right) + 10 \cdot 9 \left( \frac{10 \cdot 9}{19 \cdot 17} - \left( \frac{10}{19} \right)^2 \right) = \frac{900 \cdot 18}{(19)^2 \cdot 17} = 2.6397
\]

b) Let \( X_i = \begin{cases} 1 & \text{pair } i \text{ consists of a married couple} \\ 0 & \text{otherwise} \end{cases} \)

\[
E[X_i] = \frac{1}{19}
\]

\[
E[X_i X_j] = P(X_i = 1)P(X_j = 1|X_2 = 1) = \frac{1 \cdot 1}{19 \cdot 17}, \quad i \neq j
\]

\[
E\left[ \sum_{i=1}^{10} X_i \right] = \frac{10}{19}
\]

\[
Var\left[ \sum_{i=1}^{10} X_i \right] = 10 \cdot \frac{1}{19} \left( 1 - \frac{1}{19} \right) + 10 \cdot 9 \left( \frac{1}{19 \cdot 17} - \left( \frac{1}{19} \right)^2 \right) = \frac{180 \cdot 18}{(19)^2 \cdot 17} = 0.5280
\]

7-50. \( f_{X|p(x,y)} = \frac{e^{-\gamma} / y}{\int_0^y e^{-\gamma} / y dx} = \frac{1}{y}, \quad 0 < x < y \)

\[
E[X^3 | Y = y] = \int_0^y x^3 \frac{1}{y} dx = y^3 / 4
\]

7-57. \( E\left[ \sum_{i=1}^{n} x_i \right] = E[n]E[x] = 12.5 \)
a) \[ 6e^{-2} + .4e^{-3} \]

b) \[ .6e^{-2} \frac{2^3}{3!} + .4e^{-3} \frac{3^3}{3!} \]

c) \[ \frac{P(3|0)}{P(0)} = \frac{.6e^{-2} \frac{2^3}{3!} + .4e^{-3} \frac{3^3}{3!}}{.6e^{-2} + .4e^{-3}} \]

7-75 X is Poisson with mean \( \lambda = 2 \) and Y is binomial with parameters 10, 3/4. Hence

a) \[ P(X + Y = 2) = P(X = 0)P(Y = 2) + P(X = 1)P(Y = 1) + P(X = 2)P(Y = 0) \]
\[ = e^{-2} \binom{10}{2} \left( \frac{3}{4} \right)^2 \left( \frac{1}{4} \right)^8 + 2e^{-2} \binom{10}{1} \left( \frac{3}{4} \right) \left( \frac{1}{4} \right)^9 + 2e^{-2} \left( \frac{1}{2} \right)^{10} \]

b) \[ P(XY = 0) = P(X = 0) + P(Y = 0) - P(X = Y = 0) \]
\[ = e^{-2} + (1/4)^{10} - e^{-1}(1/4)^{10} \]

c) \[ E[XY] = E[X]E[Y] = 2 \cdot 10 \cdot \frac{3}{4} = 15 \]

7-78. a) Conditioning on the amount of the initial check gives;

\[ E[\text{Return}] = E[\text{Return}|A]/2 + E[\text{Return}|B]/2 \]
\[ = \left\{ AF(A) + B[1 - F(A)] \right\}/2 + \left\{ BF(B) + A[1 - F(B)] \right\}/2 \]
\[ = (A + B + [B - A][F(B) - F(A)])/2 \]
\[ > (A + B)/2 \]

Where the inequality follows since \([B - A] \) and \([F(B) - F(A)] \) both have the same sign.

b) If \( x < A \) then the strategy will accept the first value seen; if \( x > B \) then it will reject the first one seen; and if \( x \) lies between \( A \) and \( B \) then it will always yield return \( B \). Hence,

\[ E[\text{Return of } x- \text{ strategy}] = \begin{cases} B & \text{if } A < x < B \\ (A + B)/2 & \text{otherwise} \end{cases} \]

c) This follows from b) since there is a positive probability that \( x \) will lie between \( A \) and \( B \).

c) This follows from b) since there is a positive probability that \( x \) will lie between \( A \) and \( B \).

Chapter 7
Theoretical Exercises

7-1

Let \( \mu = E[X] \). Then for any \( a \)

\[ E[(X - a)^2] = E[(X - \mu + \mu - a)^2] \]
\[ = E[(X - \mu)^2] + (\mu - a)^2 + 2E[(X - \mu)(\mu - a)] \]
\[ = E[(X - \mu)^2] + (\mu - a)^2 + 2(\mu - a)E[(X - \mu)] \]
\[ = E[(X - \mu)^2] + (\mu - a)^2 \]
7-22. 
\[
\text{Cov}(X,Y) = b \; \text{Var}(X), \quad \text{Var}(Y) = b^2 \; \text{Var}(X)
\]
\[
\rho(X,Y) = \frac{b \; \text{Var}(X)}{\sqrt{b^2 \; \text{Var}(X)}} = \frac{b}{|b|}
\]

Chapter 8
Problems

8-1  \[ P\{0 \leq X \leq 40\} = 1 - P\{|X - 20| > 20\} \geq 1 - 20/400 = 19/20 \]

8-2. a)  \[ P\{X \geq 85\} \leq E[X]/85 = 15/17 \]

b)  \[ P\{65 \leq X \leq 85\} = 1 - P\{|X - 75| > 10\} \geq 1 - 25/100 \]

c)  \[ P\left(\sum_{i=1}^{n} X_i/n - 75 > 5\right) \leq \frac{25}{25n} \text{ so need } n = 10 \]

8-6. If \( X_i \) is the outcome of the \( i \)th roll, then \( E[X_i] = 7/2, \text{Var}(X_i) = 35/12 \) and so

\[ P\left(\sum_{i=1}^{79} X_i \leq 300\right) = P\left(\sum_{i=1}^{79} X_i \leq 300.5\right) \]
\[ = P\{N(0,1) \leq \frac{300.5 - 79(7/2)}{(79x35/12)^{1/2}}\} \]
\[ = P\{N(0,1) \leq 1.58\} = 9429 \]

8-10

If \( W_n \) is the total weight of \( n \) cars and \( A \) is the amount of weight that the bridge can withstand, then \( W_n - A \) is normal with mean value \( 3n - 400 \) and variance \( .09n + 1600 \).

Hence, the probability of structural damage is:

\[ P\{W_n - A \geq 0\} = P\{Z \geq (400 - 3n/\sqrt{.09n + 1600})\} \]

Since \( P\{Z \geq 1.28\} = .1 \) the probability of damage will exceed \( .1 \) when \( n \) is such that
400-3n \leq 1.28 \sqrt{0.9n + 1600}\}

The above will be satisfied whenever n \geq 117.

8-13.

a) 
\[ P\{ \bar{X} > 80\} = P\left\{ \frac{\bar{X} - 74}{14 \times 5} > \frac{6}{70} \right\} = P\{Z > 0.0857\} \approx 0.466 \]

b) 
\[ P\{ \bar{Y} > 80\} = P\left\{ \frac{\bar{Y} - 74}{14 \times 8} > \frac{6}{112} \right\} = P\{Z > 0.0536\} \approx 0.471 \]

c) Using that \( SD(\bar{Y} - \bar{X}) = \sqrt{196/64 + 196/25} \approx 3.30 \) we have 
\[ P\{\bar{Y} - \bar{X} > 2.2\} = P\left\{ \left| \frac{\bar{Y} - \bar{X}}{3.30} > \frac{2.2}{3.30} \right| \right\} = P\{Z > .67\} = .2514 \]

c) same as in c)

8-15 
\[ P\left\{ \sum_{i=1}^{10,000} X_i > 2,700,000\right\} \approx P\{Z \geq (2,700,000 - 2,400,000)/(800 \cdot 100) = P\{Z \geq 3.75\} \approx 0 \]

8-18

Set \( Y_i \) denote the additional number of fish that need be caught to obtain a new type when there are at present \( I \) distinct types. Then \( Y_i \) is geometric with parameter \( \frac{4-i}{4} \).

\[ E[Y] = E\left[ \sum_{i=0}^{3} Y_i \right] = 1 + \frac{4}{3} + \frac{4}{2} + 4 = \frac{25}{3} \]

\[ \text{Var}[Y] = \text{Var}\left( \sum_{i=0}^{3} Y_i \right) = \frac{4}{9} + 2 + 12 = \frac{130}{9} \]

Hence 
\[ P\{ Y - \frac{25}{3} > \sqrt{\frac{1300}{9}} \} \leq \frac{1}{10} \]

and so we can take \( a = \frac{25 - \sqrt{1300}}{3} \), and \( b = \frac{25 + \sqrt{1300}}{3} \).

\[ P\{ Y - \frac{25}{3} > a \} \leq \frac{130}{3} = 9a^2 = \frac{1}{10} \quad \text{when} \]

Also, 
\[ a = \sqrt{\frac{1170}{3}} \]. Hence, \( P\{ Y > \frac{25 + \sqrt{1170}}{3} \} \leq .1 \).
8-21  No!

8-22

a) $\frac{20}{26} \approx .769$

b) $\frac{20}{20+36} = \frac{5}{14} \approx .357$

c) $p \approx P\{Z \geq \frac{25.5-20}{\sqrt{20}}\} \approx P\{Z \geq 1.23\} \approx .1093$

d) $p = .112184$