Assignment IV
Chapter 7
Problems

7-3.

\[ E[|X - Y|] = \int_0^1 \int_0^1 (x - y) dx dy. \]

Now

\[ \int_0^1 (x - y) dx = \int_0^1 (x - y)^\alpha dy + \int_0^1 (y - x)^\alpha dy \]

\[ = \int_0^1 u^\alpha du + \int_0^{1-x} u^\alpha du \]

\[ = \frac{\alpha + 1}{2}(\alpha + 1) \]

Hence,

\[ E[|X - Y|] = \frac{1}{\alpha + 1} \int_0^1 \left( x^{\alpha + 1} + (1 - x)^{\alpha + 1} \right) dx = \frac{2}{(\alpha + 1)(\alpha + 2)}. \]

7-8.

E[number of occupied tables] = \[ \sum_{i=1}^{N} X_i \] = \[ \sum_{i=1}^{N} E[X_i] \]

Now,

\[ E[X_i] = P\{i^{th} \text{ arrival is not friends with any of first } i-1\} = (1 - p)^{i-1} \]

and so

\[ E[\text{number of occupied tables}] = \sum_{i=1}^{N} (1 - p)^{i-1} \]

7-18. E[number of matches] = \[ E\left[ \sum_{i=1}^{52} I_i \right] \]

where

\[ I_i = \begin{cases} 1 & \text{match on card } i \\ 0 & \text{no match on card } i \end{cases} \]

\[ = 52 \times \frac{1}{13} = 4, \text{ since } E[I_i] = 1/13 \]
7-39.
\[ \text{Cov}(Y_n, Y_n) = \text{Var}(Y_n) = 3\sigma^2 \]
\[ \text{Cov}(Y_n, Y_{n+1}) = \text{Cov}(X_n + X_{n+1}, X_{n+1} + X_{n+2}) = \text{Var}(X_{n+1} + X_{n+2}) = 2\sigma^2 \]
\[ \text{Cov}(Y_n, Y_{n+2}) = \text{Cov}(X_{n+2}, X_{n+2}) = \sigma^2 \]
\[ \text{Cov}(Y_n, Y_{n+j}) = 0 \text{ when } j \geq 3 \]

7-42

a) Let \( X_i = \begin{cases} 1 & \text{pair } i \text{ consists of a man and a woman} \\ 0 & \text{otherwise} \end{cases} \)

\[ E[X_i] = P[X_i = 1] = \frac{10}{19} \]

\[ E[X_i X_j] = P[X_i = 1, X_j = 1] = P[X_i = 1]P[X_j = 1 | X_2 = 1] = \frac{10 \cdot 9}{19 \cdot 17}, \quad i \neq j \]

\[ E\left[ \sum_{i=1}^{10} X_i \right] = \frac{100}{19} \]

\[ \text{Var}\left[ \sum_{i=1}^{10} X_i \right] = 10 \cdot \frac{10}{19} \left( 1 - \frac{10}{19} \right) + 10 \cdot \frac{9}{19} \cdot \frac{10}{19} \left( \frac{10}{19} \right)^2 = 900 \cdot \frac{18}{17} = 2.6397 \]

b) Let \( X_i = \begin{cases} 1 & \text{pair } i \text{ consists of a married couple} \\ 0 & \text{otherwise} \end{cases} \)

\[ E[X_i] = \frac{1}{19} \]

\[ E[X_i X_j] = P[X_i = 1]P[X_j = 1 | X_2 = 1] = \frac{1 \cdot 1}{19 \cdot 17}, \quad i \neq j \]

\[ E\left[ \sum_{i=1}^{10} X_i \right] = \frac{10}{19} \]

\[ \text{Var}\left[ \sum_{i=1}^{10} X_i \right] = 10 \cdot \frac{1}{19} \cdot \frac{15}{19} + 10 \cdot \frac{9}{19} \cdot \frac{1}{19} \cdot \left( \frac{1}{19} \right)^2 = \frac{180}{17} \cdot \frac{18}{17} = 0.5280 \]

7-50.

\[ f_{X|Y \leq y}(x) = \frac{e^{-y} / y}{\int_{0}^{y} e^{-y} / y \, dx} = \frac{1}{y}, \quad 0 < x < y \]

\[ E[X^3 | Y = y] = \int_{0}^{y} x^2 \cdot \frac{1}{y} \, dx = \frac{y^3}{4} \]
7-57. \[ E \left[ \sum_{i=1}^{n} x_i \right] = E[n]E[x] = 12.5 \]

7-68. 

a) \[ 6e^{-2} + .4e^{-3} \]

b) \[ .6e^{-2} \frac{2^3}{3!} + .4e^{-3} \frac{3^3}{3!} \]

c) \[ P\{3\mid0\} = \frac{P\{3,0\}}{P\{0\}} = \frac{.6e^{-2}e^{-2} \frac{2^3}{3!} + .4e^{-3} \frac{3^3}{3!}}{.6e^{-2} + .4e^{-3}} \]

7-78. a) Conditioning on the amount of the initial check gives:

\[ E[\text{Return}] = \frac{E[\text{Return}\mid A]}{2} + \frac{E[\text{Return}\mid B]}{2} \]

\[ = \frac{\{AF(A)+B[1-F(A)]\}}{2} + \frac{\{BF(B)+A[1-F(B)]\}}{2} \]

\[ = \frac{\{A+B+[B-A][F(B)-F(A)]\}}{2} \]

\[ \geq \frac{(A+B)}{2} \]

Where the inequality follows since \([B-A] \) and \([F(B)-F(A)] \) both have the same sign.

b) If \( x<A \) then the strategy will accept the first value seen; if \( x>B \) then it will reject the first one seen; and if \( x \) lies between \( A \) and \( B \) then it will always yield return \( B \). Hence,

\[ E[\text{Return of } x- \text{strategy}] = \begin{cases} 
B & \text{if } A<x<B \\
\frac{(A+B)}{2} & \text{otherwise}
\end{cases} \]

c) This follows from b) since there is a positive probability that \( x \) will lie between \( A \) and \( B \).

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Chapter 7

Theoretical Exercises

7-1

Let \( \mu = E[X] \). Then for any \( a \)

\[ E[(X-a)^2] = E[(X-\mu+\mu-a)^2] \]

\[ = E[(X-\mu)^2] + (\mu-a)^2 + 2E[(X-\mu)(\mu-a)] \]

\[ = E[(X-\mu)^2] + (\mu-a)^2 + 2(\mu-a)E[(X-\mu)] \]

\[ = E[(X-\mu)^2] + (\mu-a)^2 \]
7-22.
\[ \text{Cov}(X, Y) = b \ Var(X), \quad \text{Var}(Y) = b^2 \ Var(X) \]
\[ \rho(X, Y) = \frac{b \ Var(X)}{\sqrt{b^2 \ Var(X)}} = \frac{b}{|b|} \]

Chapter 8
Problems

8-1 \[ P\{0 \leq X \leq 40\} = 1 - P\{X - 20 \geq 20\} \geq 1 - 20/400 = 19/20 \]

8-2. a) \[ P\{X \geq 85\} \leq E[X]/85 = 15/17 \]

b) \[ P\{65 \leq X \leq 85\} = 1 - P\{|X - 75| > 10\} \geq 1 - 25/100 \]

c) \[ P\left\{ \left| \sum_{i=1}^{N} X_i / n - 75 \right| > 5 \right\} \leq \frac{25}{25n} \text{ so need } n = 10 \]

8-6. If \( X_i \) is the outcome of the ith roll, then \( E[X_i] = 7/2, \ Var(X_i) = 35/12 \) and so
\[ P\left\{ \sum_{i=1}^{79} X_i \leq 300 \right\} = P\left\{ \sum_{i=1}^{79} X_i \leq 300.5 \right\} \]
\[ = P\{N(0,1) \leq \frac{300.5 - 79(7/2)}{79 \times 35/12}^{1/2} \}
\[ = P\{N(0,1) \leq 1.58\} = 0.9429 \]

8-10
If \( W_n \) is the total weight of \( n \) cars and \( A \) is the amount of weight that the bridge can withstand, then \( W_n - A \) is normal with mean value \( 3n - 400 \) and variance \( .09n + 1600 \).
Hence, the probability of structural damage is;
\[ P\{W_n - A \geq 0\} \approx P\{Z \geq (400 - 3n/\sqrt{.09n + 1600}) \}
\]
Since \( P\{Z \geq 1.28\}=.1 \) the probability of damage will exceed .1 when \( n \) is such that
\[ 400 - 3n \leq 1.28 \sqrt{.09n + 1600} \]
The above will be satisfied whenever \( n \geq 117 \).
8-13.

a) 
\[ P\{\bar{X} > 80\} = P\left\{ \frac{\bar{X} - 74}{14 \times 5} > \frac{6}{70} \right\} = P\{Z > 0.0857\} \approx 0.466 \]
b) 
\[ P\{\bar{Y} > 80\} = P\left\{ \frac{\bar{Y} - 74}{14 \times 8} > \frac{6}{112} \right\} = P\{Z > 0.0536\} \approx 0.471 \]
c) Using that \( SD(\bar{Y} - \bar{X}) = \sqrt{196/64 + 196/25} = 3.30 \) we have
\[ P\{\bar{Y} - \bar{X} > 2.2\} = P\{\frac{\bar{Y} - \bar{X}}{3.30} > \frac{2.2}{3.30}\} = P\{Z > .67\} = .2514 \]

c) same as in c)

8-18

Set \( Y_i \) denote the additional number of fish that need be caught to obtain a new type when there are at present \( I \) distinct types. Then \( Y_i \) is geometric with parameter \( \frac{4 - i}{4} \).

\[
E[Y] = E\left[ \sum_{i=0}^{3} Y_i \right] = 1 + \frac{4}{3} + \frac{4}{2} + 4 = \frac{25}{3}
\]

\[
\text{Var}[Y] = \text{Var}\left( \sum_{i=0}^{3} Y_i \right) = \frac{4}{9} + 2 + 12 = \frac{130}{9}
\]

Hence
\[ P\{|Y - \frac{25}{3}| > \sqrt{\frac{130}{9}} \} \leq \frac{1}{10} \]

and so we can take \( a = \frac{25 - \sqrt{130}}{3} \), and \( b = \frac{25 + \sqrt{130}}{3} \).

\[ P\{Y > \frac{25}{3} > a\} \leq \frac{130}{130 - 9a^2} = \frac{1}{10} \]

Also,
\[ a = \sqrt{\frac{1170}{3}} \]. Hence, \( P\{Y > \frac{25 + \sqrt{1170}}{3} \leq .1 \)

8-22

a) \( 20/26 \approx .769 \)
b) \( 20/(20+36)=5/14 \approx .357 \)
c) \( p \approx P\{Z \geq (25.5 - 20)/\sqrt{20} \} \approx P\{Z \geq 1.23\} \approx .1093 \)
d) \( p = .112184 \)