UNIVERSITY OF PENNSYLVANIA
SCHOOL OF ENGINEERING AND APPLIED SCIENCE
Department of Electrical and Systems Engineering
Enm 503 Probability, Random Variables and Stochastic Processes

Make-Up Exam (Exam #1) — Fall Semester
Closed Book/Notes — November 12, 2003

Solutions

1. a 1) Given that A wins the first game, it will win the series if, from then on, it wins 2 games before B wins 3 games. Thus

$$P\{A \text{ wins } / A \text{ wins first }\} = \sum_{i=2}^{4} \binom{4}{i} p^i (1-p)^{4-1}$$

2)

$$P\{A \text{ wins first } \mid \text{A wins}\} = \frac{P\{A \text{ wins } |\text{A wins first }\} P\{A \text{ wins first}\}}{P\{A \text{ wins }\}}$$

Hence,

$$P\{A \text{ wins first } \mid \text{A wins}\} = \frac{\sum_{i=2}^{4} \binom{4}{i} p^{i+1} (1-p)^{4-i}}{\sum_{i=3}^{5} \binom{5}{i} p^{i} (1-p)^{5-i}}$$

b) The probability that a round does not result in an "odd person" is equal to 1/4, the probability that all three coins land on the same side.

a) $(1/4)^2(3/4) = 3/64$

b) $(1/4)^4 = 1/256$

2. a) If you wager x on a bet that wins the amount wagered with probability p and loses that amount with probability (1-p), then your expected winnings are

$$xp - x(1-p) = (2p-1)x$$

which is positive (and increasing in x) if and only if, $p > 1/2$. Thus, if $p \leq 1/2$ one maximizes one's expected return by wagering 0, and if $P > 1/3$ one maximizes one's expected return by wagering the maximal possible bet. Thus, if the information is that the 0.6 coin was chosen then you should bet 10, and if the information is that the 0.3 coin was chosen then you should bet 0. Hence, your expected payoff is

$$\frac{1}{2}(1.2-1)10 + \frac{1}{2}0 - C = 1 - C$$

b) Since your expected payoff is 0 without the information (because in this case the probabiity of winning is $\frac{1}{2}(0.6) + \frac{1}{2}(0.3) < \frac{1}{2}$), it follows that if the information costs less than 1 then it pays to purchase it

3. Let$D_i, i = 1, 2$ denote the event that radio i is defective. Also, let A and B be the events that the radios were produced at factory A and at factory B, respectively. Then,

$$\begin{aligned} P(D_1|D_2) &= \tfrac{P(D_1D_2)}{P(D_1)} \\ &= \tfrac{P(D_1D_2|A)P(A)+P(D_1D_2|B)P(B)}{P(D_1|A)P(A)+P(D_1|B)P(B)} \\ &= \tfrac{(.05)^2(1/2)+(.01)^2(1/2)}{(.05)(1/2)+(.01)(1/2)} \\ &= \tfrac{13}{300} \end{aligned}$$

4. a) Here we let $x_i, i = 1, 2, 3, 4, 5.$, denote the i^{th} art collector. Then we have the (number of solutions for selling all the Dalis is $x_1 + \ldots + x_5 = 4$) and the (number of solutions for selling the van Goghs is $(x_1 + \cdots + x_5 = 5)$) and the (number of solutions for selling the Picassos is $x_1 \ldots + x_5 = 6$). Now, we need to multiply them together to get

$$\binom{8}{4}\binom{9}{4}\binom{10}{4} = 185,220$$

b) There are $9 \cdot 8 \cdot 7 \cdot 6 \cdot 5$ numbers in which no digit is repeated. There ar $\binom{5}{2} \cdot 8 \cdot 7 \cdot 6$ numbers in which only one specified digit appears twice, and so there are $9\binom{5}{2} \cdot 8 \cdot 7 \cdot 6$ numbers in which only a single digit appears twice. There are $7 \cdot \frac{5!}{2!2!}$ numbers in which two specified digits appear twice, so there are $\binom{9}{2} 7 \cdot \frac{5!}{2!2!}$ numbers in which two digits appear twice. Thus, the answer is

$$9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 + 9\binom{5}{2} \cdot 8 \cdot 7 \cdot 6 + \binom{9}{2} 7 \cdot \frac{5!}{2!2!} = 53,220$$

5. There are $(10)!/2^5$ different divisions of the 10 players into a first roommate pair, a second roommate pair, and so on. Hence, there are $(10)!/(5!2^5)$ divisions into 5 roommate pairs. There are $\binom{6}{2}\binom{4}{2}$ ways of choosing the frontcourt and backcourt players to be in the mixed roommate pairs, and then 2 ways of pairing them up. As there is then 1 way to pair up the remaining two backcourt, and $4!/(2!2^2) = 3$ ways of making two roommate pairs from the remianing four frontcourt players, we see that

the desired probability is

$$P\{2 \text{ mixed pairs }\} = \frac{\binom{6}{2}\binom{4}{2}(2)(3)}{(10)!/(5!2^5)} = 0.5714$$