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Frobenius Solution to a 2nd order ODE near a regular singular point

Consider the ODE

$$y''(x)+P(x)\,y'(x)+Q(x)\,y(x)=0 \tag{1}$$

We will look for series solutions to (1) around (at most) *regular singular points*, which without loss in generality will be located at $x=0$. The notation adopted below closely follows that in the notes of Carchidi (handout and on web site) and in Hildebrand (Sec. 4.4 – with his $R(x)=1$ multiplying y'').

If functions $P(x)$ and $Q(x)$ are *regular* around $x=0$, i.e. if they possess a Taylor series expansion around $x=0$, then $y(x)$ can also be expressed in a Taylor series around $x=0$. Substituting the Taylor series for $P(x)$, $Q(x)$ and $y(x)$ into (1) and requiring that each term of like powers of x sums to zero (each coefficient of a Taylor series for zero is zero) leads to the unknown coefficients of the series for $y(x)$. The method of Frobenius is an extension of this idea to equations with regular singular points and builds on what we know about equidimensional equations (see handout on equidimensional equations).

If (1) has a *regular singular point* at $x=0$, then the following limits exist:

$$p_o=\lim_{x\to 0}\left[x\,P(x)\right] \text{ and } q_o=\lim_{x\to 0}\left[x^2\,Q(x)\right] \tag{2}$$

Note that the existence of these limits implies that (1) can be rewritten in the form:

$$y''(x)+\frac{\hat{P}(x)}{x}y'(x)+\frac{\hat{Q}(x)}{x^2}y(x)=0 \tag{3}$$

where $\hat{P}(x)$ and $\hat{Q}(x)$ are regular around $x=0$, in which case (3) is a generalization of an equidimensional equation. Given this latter observation and what we know about Taylor series solutions about regular points, we look for Frobenius series solutions (1) or (3):

$$y(x) = y(x;s) = x^s \sum_{n=0}^{\infty} a_n \, x^n = \sum_{n=0}^{\infty} a_n \, x^{n+s} \tag{4}$$

The exponent s is determined from the (quadratic since (1) is a 2^{nd} order ODE) *indicial equation*:

$$f(s) = s^2 + (p_o - 1)s + q_o = 0 \tag{5}$$

which leads to two values of s. By convention, we order s_1 and s_2 such that $\operatorname{Re}(s_1) > \operatorname{Re}(s_2)$. In principle, each root gives a Frobenius series (we must be careful about special cases just as with equidimensional equations). The coefficients of those series are given in equations (2.9) with (2.7b) of Carchidi's notes, where p_n and q_n are the Taylor coefficients of the respective series for $x\,P(x)$ and $x^2\,Q(x)$. Next we summarize 3 special cases before we take advantage of Maple to do the tedious algebra.

One solution to (1) can always be expressed in the form of (4):

$$y_1(x) = x^{s_1} \sum_{n=0}^{\infty} a_n \, x^n = \sum_{n=0}^{\infty} a_n \, x^{n+s_1} \tag{6}$$

The second solution is determined according the following 3 cases:

Case 1: Consider $s_1 - s_2$ **not an integer** (including when s_1 and s_2 are complex conjugates – case 1 includes Carchidi's cases a) and c):

$$y_2(x) = x^{s_2} \sum_{n=0}^{\infty} b_n \, x^n = \sum_{n=0}^{\infty} b_n \, x^{n+s_2} \tag{7.1}$$

Case 2: Consider $s_1 = s_2 \equiv s_o$ (Carchidi's case b):

$$y_2(x) = y_1(x)\ln(x) + \sum_{n=0}^{\infty} b_n \, x^{n+s_o} \tag{7.2}$$

Case 3: Consider $s_1 - s_2$ **is a positive integer** (Carchidi's case b):

$$y_2(x) = c\,y_1(x)\ln(x) + \sum_{n=0}^{\infty} b_n \, x^{n+s_2} \tag{7.3}$$

where the constant c can be zero or nonzero.

The strategy to solve a particular problem is to first determine s_1 and s_2 from (5) and go on to find the coefficients a_n for $y_1(x)$in (6) by summing the coefficients of like powers of x to zero. The second solution is found in a similar way using the form for $y_2(x)$ given by cases 1, 2 or 3. Two (handwritten) examples appear below. One can also use Maple. For example, to determine Frobenius solutions to Bessel's equation of zeroth order the following code can be used:

```
#  Define  Bessel's equation of zeroth order:
> L(u):=x^2*diff(u(x),x$2)+x*diff(u(x),x)+(a^2*x^2)*u(x);
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$$\mathrm{L}(u) := x^2 \left(\frac{\partial^2}{\partial x^2}\,\mathrm{u}(x)\right) + x\left(\frac{\partial}{\partial x}\,\mathrm{u}(x)\right) + a^2\,x^2\,\mathrm{u}(x)$$

```
> assume(a, real, a>0);
#  Obtain a series solution (for J(0,ax) and Y(0,ax)):
> Order:= 7: dsolve({L(u)=0},u(x),series);
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$$\mathrm{u}(x) = \_C1\left(1 - \frac{1}{4}a{\sim}^2\,x^2 + \frac{1}{64}a{\sim}^4\,x^4 - \frac{1}{2304}a{\sim}^6\,x^6 + \mathrm{O}(x^7)\right) + \_C2\Bigg(\ln(x)\left(1 - \frac{1}{4}a{\sim}^2\,x^2 + \frac{1}{64}a{\sim}^4\,x^4 - \frac{1}{2304}a{\sim}^6\,x^6 + \mathrm{O}(x^7)\right) + \left(\frac{1}{4}a{\sim}^2\,x^2 - \frac{3}{128}a{\sim}^4\,x^4 + \frac{11}{13824}a{\sim}^6\,x^6 + \mathrm{O}(x^7)\right)\Bigg)$$

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#  Or, Maple can solve the equation symbolically:
> sol:=dsolve({L(u)=0});
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$$sol := \mathrm{u}(x) = \_C1\ \mathrm{BesselJ}(0, a{\sim}\,x) + \_C2\ \mathrm{BesselY}(0, a{\sim}\,x)$$

Although Maple has done tedious algebra for us in determining the Frobenius series, one still should determine the pattern in the coefficients, i.e. the recursion relation. For example, the series for $J_n(x)$ can be expressed compactly as:

$$\mathrm{J}_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{2k+n}}{k!\,(k+n)!}$$

Handout of 1/29/01 (note: for the equidimensional equation only $s_1 = s_2 \equiv s_o$ is a special case)

Equidimensional ODEs (Euler equations) have a **regular singular point**, which in the following examples is located at $x_o = 0$, i.e. $u \sim x^s$ (see pp. 142-143 in Wylie and Barrett – more on singular points when we discuss the method of **Frobenius**). Since the exponent s can be negative, the solution can be singular at $x = 0$. Furthermore, when an exponent s is a repeated root of the **characteristic equation**, solutions also exist of the form $u \sim \left[\ln(s)\right]^m x^s$, $m = 1, 2, \ldots, M-1$, where M is the multiplicity of the root. Examples for various cases follow.

1) $L(u) = x^2 u'' - 2xu' + 2u = 0$

$$\begin{aligned} L\left(x^s\right) &= \left[s(s-1) - 2s + 2\right] x^s \\ &= \left(s^2 - 3s + 2\right) x^s \\ &= (s-1)(s-2)\, x^s \end{aligned}$$

The characteristic equation $s^2 - 3s + 2 = 0$ has roots $s_1 = 1$ and $s_2 = 2$.

$$\Rightarrow \quad u(x) = c_1 x + c_2 x^2 \qquad \text{(no singularity at } x = 0\text{)}$$

2) $L(u) = 2x^2 u'' + 5xu' + u = 0$

$$\begin{aligned} L\left(x^s\right) &= \left[2s(s-1) + 5s + 1\right] x^s \\ &= \left(2s^2 + 3s + 1\right) x^s \\ &= (2s+1)(s+1)\, x^s \end{aligned}$$

$$\Rightarrow \quad u(x) = c_1 x^{-1/2} + c_2 x^{-1} \qquad \text{(singularity at } x = 0\text{)}$$

3) $L(u) = x^2 u'' - x u' + u = 0$

$$L(x^s) = [s(s-1) - s + 1]\, x^s$$
$$= (s^2 - 2s + 1)\, x^s$$
$$= (s-1)^2\, x^s$$

Hence, $s = 1$ is a double root ($M = 2$). Note that:

$$\frac{\partial}{\partial s} L(x^s) = 2(s-1)x^s + (s-1)^2 (\ln x)\, x^s$$
$$= 0 \quad \text{if } s = 1$$

Since $\frac{\partial}{\partial s} L(x^s) = L\left(\frac{\partial x^s}{\partial s}\right)$ it follows that

$\left.\frac{\partial x^s}{\partial s}\right|_{s=1} = x \ln x$ is also a solution.

The complete solution is: $u(x) = c_1 x + c_2 x \ln x$

To see that this solution is not singular at $x = 0$ note that

$$\lim_{x\to 0} x^s \ln x = \lim_{x\to 0} \frac{\ln x}{x^{-s}}$$
$$= \lim_{x\to 0} \frac{1/x}{-s\, x^{-s-1}}$$
$$= \lim_{x\to 0} -\frac{x^s}{s}$$
$$= 0 \quad \text{if } s > 0$$