THE METHODS OF TAYLOR AND FROBENIUS

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I.) THE POWER SERIES METHOD, EXPANSION ABOUT AN ORDINARY POINT

We have seen in class that $x = x_o$ is an *ordinary* point for the differential equation

$$y''(x) + P(x)y'(x) + Q(x)y(x) = 0 \tag{1.1}$$

if

$$\lim_{x\to x_o} P(x) \qquad \text{and} \qquad \lim_{x\to x_o} Q(x) \tag{1.2}$$

both exist. Since the limits in Equation (1.2) allow us to write

$$P(x) = \sum_{n=0}^{\infty} p_n(x-x_o)^n \tag{1.3a}$$

and

$$Q(x) = \sum_{n=0}^{\infty} q_n(x-x_o)^n, \tag{1.3b}$$

we may expand the solution to Equation (1.1) as

$$y(x) = \sum_{n=0}^{\infty} a_n(x-x_o)^n. \tag{1.4a}$$

Differentiating this result leads to,

$$y'(x) = \sum_{n=0}^{\infty} na_n(x-x_o)^{n-1} = 0 + \sum_{n=1}^{\infty} na_n(x-x_o)^{n-1}$$

or (after shifting the summation index by one unit down),

$$y'(x) = \sum_{n=0}^{\infty}(n+1)a_{n+1}(x-x_o)^n, \tag{1.4b}$$

while a further differentiation leads to

$$y''(x) = \sum_{n=0}^{\infty} n(n+1)a_{n+1}(x-x_o)^{n-1} = 0 + \sum_{n=1}^{\infty} n(n+1)a_{n+1}(x-x_o)^{n-1}$$

or (after shifting the summation index by one unit down),

$$y''(x) = \sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}(x-x_o)^n. \tag{1.4c}$$

If we operate on Equation (1.4a) with L, where

$$L = \frac{d^2}{dx^2} + P(x)\frac{d}{dx} + Q(x),$$

then

$$\begin{aligned} L[y] &= \sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}(x-x_o)^n + P(x)\sum_{n=0}^{\infty}(n+1)a_{n+1}(x-x_o)^n \\ &\quad + Q(x)\sum_{n=0}^{\infty} a_n(x-x_o)^n. \end{aligned}$$

If we put in Equations (1.3a) and (1.3b) for $P(x)$ and $Q(x)$, respectively, we get

$$\begin{aligned} L[y] &= \sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}(x-x_o)^n \\ &\quad + \left(\sum_{n=0}^{\infty} p_n(x-x_o)^n\right)\left(\sum_{n=0}^{\infty}(n+1)a_{n+1}(x-x_o)^n\right) \\ &\quad + \left(\sum_{n=0}^{\infty} q_n(x-x_o)^n\right)\left(\sum_{n=0}^{\infty} a_n(x-x_o)^n\right). \end{aligned}$$

Using the algebraic identity,

$$\left(\sum_{n=0}^{\infty} A_n\right)\left(\sum_{n=0}^{\infty} B_n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} A_{n-k}B_k\right), \tag{1.5}$$

we may write for $L[y]$ above,

$$\begin{aligned}
L[y] &= \sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}(x-x_o)^n \\
&\quad + \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} p_{n-k}(x-x_o)^{n-k}(k+1)a_{k+1}(x-x_o)^k\right) \\
&\quad + \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} q_{n-k}(x-x_o)^{n-k}a_k(x-x_o)^k\right) \\
&= \sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}(x-x_o)^n \\
&\quad + \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} p_{n-k}(k+1)a_{k+1}\right)(x-x_o)^n \\
&\quad + \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} q_{n-k}a_k\right)(x-x_o)^n
\end{aligned}$$

or

$$L[y] = \sum_{n=0}^{\infty}\left\{(n+1)(n+2)a_{n+2} + \sum_{k=0}^{n}\left((k+1)p_{n-k}a_{k+1} + q_{n-k}a_k\right)\right\}(x-x_o)^n.$$

Setting $L[y]=0$, then yields

$$\sum_{n=0}^{\infty}\left\{(n+1)(n+2)a_{n+2} + \sum_{k=0}^{n}\left((k+1)p_{n-k}a_{k+1} + q_{n-k}a_k\right)\right\}(x-x_o)^n = 0$$

for all x in some region $|x-x_o|<R$, which means,

$$(n+1)(n+2)a_{n+2} + \sum_{k=0}^{n}\left((k+1)p_{n-k}a_{k+1} + q_{n-k}a_k\right) = 0 \tag{1.6}$$

for $n = 0, 1, 2, \ldots$. Solving this for a_{n+2} then leads to

$$a_{n+2} = \frac{-\sum\limits_{k=0}^{n}\left((k+1)p_{n-k}a_{k+1} + q_{n-k}a_k\right)}{(n+1)(n+2)} \tag{1.7}$$

or

$$a_{n+2} = -\left\{\frac{q_n}{(n+1)(n+2)}\right\}a_0 - \sum_{k=1}^{n}\left\{\frac{kp_{n-k+1} + q_{n-k}}{(n+1)(n+2)}\right\}a_k - \left\{\frac{p_0}{n+2}\right\}a_{n+1} \tag{1.8}$$

for $n = 0, 1, 2, \ldots$. This says, for example, that

$$a_2 = -\left\{\frac{q_0}{2}\right\} a_0 - \left\{\frac{p_0}{2}\right\} a_1$$

and

$$a_3 = -\left\{\frac{q_1}{6}\right\} a_0 - \left\{\frac{p_1 + q_0}{6}\right\} a_1 - \left\{\frac{p_0}{3}\right\} a_2$$

or

$$a_3 = -\left\{\frac{q_1}{6}\right\} a_0 - \left\{\frac{p_1 + q_0}{6}\right\} a_1 - \left\{\frac{p_0}{3}\right\} \left\{-\left\{\frac{q_0}{2}\right\} a_o - \left\{\frac{p_0}{2}\right\} a_1\right\}$$

$\Longrightarrow$

$$a_3 = \left\{\frac{p_0 q_0 - q_1}{6}\right\} a_0 + \left\{\frac{p_0^2 - p_1 - q_0}{6}\right\} a_1.$$

Continuing this process leads to fact that (in general) a_n can be expressed as a linear combination of a_0 and a_1. Specifically, we may write

$$a_n = A_n(p_0, p_1, \ldots, p_n, q_0, q_1, \ldots, q_n)a_0 + B_n(p_0, p_1, \ldots, p_n, q_0, q_1, \ldots, q_n)a_1 \quad (1.9)$$

for $n = 2, 3, 4, \ldots$, where $A_n(p_0, p_1, \ldots, p_n, q_0, q_1, \ldots, q_n)$ and $B_n(p_0, p_1, \ldots, p_n, q_0, q_1, \ldots, q_n)$ are functions of $p_0, p_1, \ldots, p_n$, and $q_0, q_1, \ldots, q_n$. Rewriting Equation (1.4a) as

$$y(x) = a_0 + a_1(x - x_o) + \sum_{n=2}^{\infty} a_n(x - x_o)^n$$

and putting in Equation (1.9) then gives

$$\begin{aligned} y(x) &= a_0 + a_1(x - x_o) + \sum_{n=2}^{\infty} \{A_n(p_0, p_1, \ldots, p_n, q_0, q_1, \ldots, q_n)a_0 \\ &\quad + B_n(p_0, p_1, \ldots, p_n, q_0, q_1, \ldots, q_n)a_1\} (x - x_o)^n \end{aligned}$$

or

$$\begin{aligned} y(x) &= a_0 \left\{1 + \sum_{n=2}^{\infty} A_n(p_0, p_1, \ldots, p_n, q_o, q_1, \ldots, q_n)(x - x_o)^n\right\} \\ &\quad + a_1 \left\{(x - x_o) + \sum_{n=2}^{\infty} B_n(p_0, p_1, \ldots, p_n, q_o, q_1, \ldots, q_n)(x - x_o)^n\right\} \end{aligned}$$

where a_0 and a_1 are arbitrary. This yields for the *general* solution of Equation (1.1),

$$y(x) = a_0 y_1(x) + a_1 y_2(x) \quad (1.10)$$

with

$$y_1(x) = 1 + \sum_{n=2}^{\infty} A_n(p_0, p_1, \ldots, p_n, q_0, q_1, \ldots, q_n)(x - x_o)^n \tag{1.11a}$$

and

$$y_2(x) = (x - x_o) + \sum_{n=2}^{\infty} B_n(p_0, p_1, \ldots, p_n, q_0, q_1, \ldots, q_n)(x - x_o)^n. \tag{1.11b }$$

Therefore, a series expansion about an ordinary point, in principle, will always lead to both solutions $y_1(x)$ and $y_2(x)$, and these solutions will be well-behaved in a region $|x - x_o| < R$.

II.) THE POWER SERIES METHOD, EXPANSION ABOUT A REGULAR SINGULAR POINT

We have seen in class that $x = x_o$ is a *regular singular* point for the differential equation

$$y''(x) + P(x)y'(x) + Q(x)y(x) = 0 \tag{2.1}$$

if either

$$\lim_{x\to x_o} P(x) \qquad \text{or} \qquad \lim_{x\to x_o} Q(x)$$

does not exist, yet

$$\lim_{x\to x_o} (x - x_o)P(x) \qquad \text{and} \qquad \lim_{x\to x_o} (x - x_o)^2 Q(x) \tag{2.2}$$

both do exist. Since the limits in Equation (2.2) allow us to write

$$(x - x_o)P(x) = \sum_{n=0}^{\infty} p_n (x - x_o)^n \tag{2.3a}$$

and

$$(x - x_o)^2 Q(x) = \sum_{n=0}^{\infty} q_n (x - x_o)^n, \tag{2.3b}$$

or,

$$P(x) = (x - x_o)^{-1} \sum_{n=0}^{\infty} p_n (x - x_o)^n \tag{2.3c}$$

and

$$Q(x) = (x - x_o)^{-2} \sum_{n=0}^{\infty} q_n (x - x_o)^n, \tag{2.3d}$$

we may expand the solution to Equation (2.1) as

$$y(x,s)=\sum_{n=0}^{\infty}a_n(s)(x-x_o)^{s+n}, \tag{2.4a}$$

where $a_0(s)\neq 0$, and s is a parameter to be determined.

Differentiating Equation (2.4a) with respect to x leads to,

$$y'(x,s)=\sum_{n=0}^{\infty}(s+n)a_n(s)(x-x_o)^{s+n-1}. \tag{2.4b}$$

while a further differentiation with respect to x leads to

$$y''(x,s)=\sum_{n=0}^{\infty}(s+n)(s+n-1)a_n(x-x_o)^{s+n-2}. \tag{2.4c}$$

If we operate on Equation (2.4a) with L, where

$$L=\frac{d^2}{dx^2}+P(x)\frac{d}{dx}+Q(x),$$

then

$$\begin{aligned}L[y(x,s)] &= \sum_{n=0}^{\infty}(s+n)(s+n-1)a_n(s)(x-x_o)^{s+n-2}\\ &\quad +P(x)\sum_{n=0}^{\infty}(s+n)a_n(s)(x-x_o)^{s+n-1}\\ &\quad +Q(x)\sum_{n=0}^{\infty}a_n(s)(x-x_o)^{s+n}.\end{aligned}$$

If we put in Equations (2.3c) and (2.3d) for $P(x)$ and $Q(x)$, we get

$$\begin{aligned}L[y(x,s)] &= \sum_{n=0}^{\infty}(s+n)(s+n-1)a_n(s)(x-x_o)^{s+n-2}\\ &\quad +\left((x-x_o)^{-1}\sum_{n=0}^{\infty}p_n(x-x_o)^n\right)\left(\sum_{n=0}^{\infty}(s+n)a_n(s)(x-x_o)^{s+n-1}\right)\\ &\quad +\left((x-x_o)^{-2}\sum_{n=0}^{\infty}q_n(x-x_o)^n\right)\left(\sum_{n=0}^{\infty}a_n(s)(x-x_o)^{s+n}\right).\end{aligned}$$

Using the algebraic identity,

$$\left(\sum_{n=0}^{\infty} A_n\right)\left(\sum_{n=0}^{\infty} B_n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} A_{n-k}B_k\right), \tag{2.5}$$

we may write for $L[y]$ above,

$$\begin{aligned} L[y(x,s)] &= \sum_{n=0}^{\infty}(s+n)(s+n-1)a_n(s)(x-x_o)^{s+n-2} \\ &\quad +(x-x_o)^{-1}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} p_{n-k}(x-x_o)^{n-k}(s+k)a_k(s)(x-x_o)^{s+k-1}\right) \\ &\quad +(x-x_o)^{-2}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} q_{n-k}(x-x_o)^{n-k}a_k(s)(x-x_o)^{s+k}\right) \\ &= \sum_{n=0}^{\infty}(s+n)(s+n-1)a_n(s)(x-x_o)^{s+n-2} \\ &\quad +\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(s+k)p_{n-k}a_k(s)\right)(x-x_o)^{s+n-2} \\ &\quad +\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} q_{n-k}a_k(s)\right)(x-x_o)^{s+n-2} \end{aligned}$$

$\Longrightarrow$

$$\begin{aligned} L[y(x,s)] &= \sum_{n=0}^{\infty}\Big\{(s+n)(s+n-1)a_n(s) \\ &\qquad +\sum_{k=0}^{n}((s+k)p_{n-k}+q_{n-k})a_k(s)\Big\}(x-x_o)^{s+n-2}. \end{aligned}$$

or

$$\begin{aligned} L[y(x,s)] &= \{s(s-1)+p_0 s+q_0\}\, a_0(s)(x-x_o)^{s-2} \\ &+\sum_{n=1}^{\infty}\Big\{[(s+n)(s+n-1)+p_0(s+n)+q_0]\, a_n(s) \\ &+\sum_{k=0}^{n-1}((s+k)p_{n-k}+q_{n-k})a_k(s)\Big\}(x-x_o)^{s+n-2}. \end{aligned} \tag{2.6}$$

To simplify the notation, we define the *quadratic* polynomial is s,

$$f(s) = s^2+(p_0-1)s+q_0 \tag{2.7a}$$

and the set of *linear* polynomials in s,

$$\gamma_{n,k}(s) = (s+k)p_{n-k} + q_{n-k} \tag{2.7b}$$

so that Equation (2.6) becomes

$$\begin{aligned} L[y(x,s)] &= f(s)a_0(s)(x-x_o)^{s-2} \\ &+ \sum_{n=1}^{\infty}\left\{ f(s+n)a_n(s) + \sum_{k=0}^{n-1}\gamma_{n,k}(s)a_k(s)\right\}(x-x_o)^{s+n-2}. \end{aligned} \tag{2.8}$$

If we now require that

$$f(s+n)a_n(s) + \sum_{k=0}^{n-1}\gamma_{n,k}(s)a_k(s) = 0 \tag{2.9}$$

for $n = 1, 2, 3, \ldots$, then Equation (2.8) reads

$$L[y(x,s)] = f(s)a_0(s)(x-x_o)^{s-2}. \tag{2.10}$$

This says that $y(x,s)$ is a solution to $L[y(x,s)] = 0$, if s is chosen so that

$$f(s) = s^2 + (p_0 - 1)s + q_0 = 0.$$

because, $a_0(s) \neq 0$. This leads to

$$f(s) = (s - s_1)(s - s_2) = 0 \tag{2.11}$$

where

$$s_1 = \frac{1}{2}\left\{-(p_0-1) + \sqrt{(p_0-1)^2 - 4q_0}\right\} \tag{2.12a}$$

and

$$s_2 = \frac{1}{2}\left\{-(p_0-1) - \sqrt{(p_0-1)^2 - 4q_0}\right\} \tag{2.12b}$$

with

$$s_1 - s_2 = \sqrt{(p_0-1)^2 - 4q_0} \geq 0$$

when both are real numbers. Therefore

$$L[y(x,s)] = (s-s_1)(s-s_2)a_0(s)(x-x_o)^{s-2}. \tag{2.13}$$

and hence

$$L[y(x,s_1)] = L[y(x,s_2)] = 0. \tag{2.14}$$

Equation (2.14) says that $y(x,s_1)$ and $y(x,s_2)$ are solutions to Equation (2.1) provided, of course that the assumptions in Equation (2.9) are valid, and these assumptions are valid as long as it is possible to solve for the $a_n(s)'s$.

To see if we can always solve for the $a_n(s)'s$ let us first rewrite Equation (2.9) as

$$(s+n-s_1)(s+n-s_2)a_n(s) + \sum_{k=0}^{n-1} \gamma_{n,k}(s)a_k(s) = 0$$

or

$$(s+n-s_1)(s+n-s_2)a_n(s) = -\sum_{k=0}^{n-1} \gamma_{n,k}(s)a_k(s) \tag{2.15}$$

for $n = 1, 2, 3, \ldots$, and now let us consider some cases.

a.) THE CASE WHEN $(p_0 - 1)^2 - 4q_0 < 0$

Under this case both s_1 and s_2 are *complex.* If we set $s = s_1$ in Equation (2.15), then

$$n(n+s_1-s_2)a_n(s_1) = -\sum_{k=0}^{n-1} \gamma_{n,k}(s_1)a_k(s_1)$$

$\Longrightarrow$

$$n\left\{n + \sqrt{(p_0-1)^2 - 4q_0}\right\} a_n(s_1) = -\sum_{k=0}^{n-1} \gamma_{n,k}(s_1)a_k(s_1)$$

or

$$n\left\{n + i\sqrt{4q_0 - (p_0-1)^2}\right\} a_n(s_1) = -\sum_{k=0}^{n-1} \gamma_{n,k}(s_1)a_k(s_1)$$

for $n = 1, 2, 3, \ldots$. But

$$n\left\{n + i\sqrt{4q_0 - (p_0-1)^2}\right\} \neq 0$$

or all possible integer values of n. Therefore, we may divide by this and write

$$a_n(s_1) = \frac{-\sum\limits_{k=0}^{n-1} \gamma_{n,k}(s_1)a_k(s_1)}{n\left\{n + i\sqrt{4q_0 - (p_0-1)^2}\right\}}$$

for $n = 1, 2, 3, \ldots$, which means that we can *always solve* for all $a_n(s_1)'s$ for $n = 1, 2, 3, \ldots$, in terms of just $a_0(s_1)$. A similar conclusion is reached if we make $s = s_2$ in Equation (2.15). As a result, both

$$y_1(x) = y(x, s_1) = \sum_{n=0}^{\infty} a_n(s_1)(x - x_o)^{s_1+n} \tag{2.16a}$$

and

$$y_2(x) = y(x, s_2) = \sum_{n=0}^{\infty} a_n(s_2)(x - x_o)^{s_2+n} \tag{2.16b}$$

are linearly independent solutions to Equation (2.1), and

$$y(x) = c_1 y(x, s_1) + c_2 y(x, s_2)$$

is the general solution to Equation (2.1).

b.) THE CASE WHEN $(p_0 - 1)^2 - 4q_0 = 0$

For this case we have for Equation (2.12a) and (2.12b),

$$s_1 = s_2 = \frac{1 - p_0}{2} \equiv s_o \tag{2.17}$$

which is a *real* number. Putting this into Equation (2.15) leads to

$$(s + n - s_o)(s + n - s_o)a_n(s) = -\sum_{k=0}^{n-1} \gamma_{n,k}(s)a_k(s)$$

which (after setting $s = s_o$) reduces to

$$n^2 a_n(s_o) = -\sum_{k=0}^{n-1} \gamma_{n,k}(s_o)a_k(s_o)$$

for $n = 1, 2, 3, \ldots$, and so

$$a_n(s_o) = \frac{-\sum\limits_{k=0}^{n-1} \gamma_{n,k}(s_o)a_k(s_o)}{n^2} \tag{2.18}$$

which means we can always solve for the $a_n(s_o)'s$ in terms of $a_0(s_o)$, and so

$$y_1(x) = y(x, s_o) = \sum_{n=0}^{\infty} a_n(s_o)(x - x_o)^{s_o+n} \tag{2.19}$$

is always one solution to Equation (2.1). A second solution can then be obtained using

$$y_2(x) = y_1(x) \int \left\{ \frac{e^{-\int P(x)dx}}{y_1^2(x)} \right\} dx \tag{2.20}$$

or by the following procedure.

If we take Equation (2.13) and set $s_1 = s_2 = s_o$, then

$$L[y(x,s)] = (s - s_o)^2 a_0(s)(x - x_o)^{s-2}. \tag{2.21}$$

If we differentiate Equation (2.21) *with respect to* s, then

$$\begin{aligned}
\frac{\partial}{\partial s} L[y(x,s)] &= \frac{\partial}{\partial s}\left((s-s_o)^2 a_0(s)(x-x_o)^{s-2}\right) \\
&= \frac{\partial (s-s_o)^2}{\partial s} a_0(s)(x-x_o)^{s-2} + (s-s_o)^2 \frac{\partial a_0(s)}{\partial s}(x-x_o)^{s-2} \\
&\quad +(s-s_o)^2 a_0(s) \frac{\partial (x-x_o)^{s-2}}{\partial s} \\
&= 2(s-s_o)a_0(s)(x-x_o)^{s-2} + (s-s_o)^2 a_0'(s)(x-x_o)^{s-2} \\
&\quad +(s-s_o)^2 a_0(s) \ln|x-x_o|(x-x_o)^{s-2}.
\end{aligned}$$

Recall that

$$\frac{d}{dx} a^x = (\ln|a|)a^x. \tag{2.22}$$

In addition, since L operates on functions x and $\frac{\partial}{\partial s}$ operates on functions of s, (with x and s independent) we may invert the order and write,Therefore

$$\begin{aligned}
L\left[\frac{\partial}{\partial s} y(x,s)\right] &= 2(s-s_o)a_0(s)(x-x_o)^{s-2} + (s-s_o)^2 a_0'(s)(x-x_o)^{s-2} \\
&\quad +(s-s_o)^2 a_0(s) \ln|x-x_o|(x-x_o)^{s-2},
\end{aligned}$$

which leads to

$$L\left[\lim_{s\to s_o} \frac{\partial y(x,s)}{\partial s}\right] = 0.$$

Therefore

$$y_2(x) = \lim_{s\to s_o} \frac{\partial y(x,s)}{\partial s} \tag{2.23}$$

is a second linearly independent solution to Equation (2.1).

Note that

$$y(x,s)=\sum_{n=0}^{\infty}a_n(s)(x-x_o)^{s+n}$$

so that

$$\begin{aligned}\frac{\partial y(x,s)}{\partial s} &= \sum_{n=0}^{\infty}\frac{\partial}{\partial s}\Big(a_n(s)(x-x_o)^{s+n}\Big)\\ &= \sum_{n=0}^{\infty}\left\{a_n'(s)(x-x_o)^{s+n}+a_n(s)\frac{\partial}{\partial s}(x-x_o)^{s+n}\right\}\\ &= \sum_{n=0}^{\infty}\Big\{a_n'(s)(x-x_o)^{s+n}+a_n(s)(x-x_o)^{s+n}\ln|x-x_o|\Big\}\\ &= \sum_{n=0}^{\infty}a_n'(s)(x-x_o)^{s+n}+\ln|x-x_o|\sum_{n=0}^{\infty}a_n(s)(x-x_o)^{s+n}.\end{aligned}$$

If

$$y_1(x)=y(x,s_o)=\sum_{n=0}^{\infty}a_n(s_o)(x-x_o)^{s_o+n}, \tag{2.24a}$$

then

$$y_2(x)=\lim_{s\to s_o}\frac{\partial y(x,s)}{\partial s}=\sum_{n=0}^{\infty}a_n'(s_o)(x-x_o)^{s_o+n}+\ln|x-x_o|\sum_{n=0}^{\infty}a_n(s_o)(x-x_o)^{s_o+n}$$

or

$$y_2(x)=\lim_{s\to s_o}\frac{\partial y(x,s)}{\partial s}=\sum_{n=0}^{\infty}a_n'(s_o)(x-x_o)^{s_o+n}+y_1(x)\ln|x-x_o|. \tag{2.24b}$$

Note then that the general solution to Equation (2.1) now reads

$$y(x)=c_1y_1(x)+c_2y_2(x) \tag{2.25}$$

with $y_1(x)$ and $y_2(x)$ given in Equations (2.24a) and (2.24b). Therefore this case always leads to a general solution.

c.) THE CASE WHEN $(p_0 - 1)^2 - 4q_0 > 0$ AND $s_1 - s_2$ ***IS NOT*** A POSITIVE INTEGER

This case is very similar to case (a) above except that under this case both s_1 and s_2 are *real.* However, if we set $s = s_1$ in Equation (2.15), then

$$n(n + s_1 - s_2)a_n(s_1) = -\sum_{k=0}^{n-1} \gamma_{n,k}(s_1)a_k(s_1)$$

for $n = 1, 2, 3, \ldots$. But $s_1 - s_2 \neq$ *integer* means that

$$n(n + s_1 - s_2) \neq 0$$

or all possible *integer* values of n. Therefore, we may divide by this and write

$$a_n(s_1) = \frac{-\sum\limits_{k=0}^{n-1} \gamma_{n,k}(s_1)a_k(s_1)}{n(n + s_1 - s_2)}$$

for $n = 1, 2, 3, \ldots$, which means that we can *always solve* for all $a_n(s_1)'s$ for $n = 1, 2, 3, \ldots$, in terms of just $a_0(s_1)$. A similar conclusion is reached if we make $s = s_2$ in Equation (2.15). As a result, both

$$y_1(x) = y(x, s_1) = \sum_{n=0}^{\infty} a_n(s_1)(x - x_o)^{s_1+n} \tag{2.26a}$$

and

$$y_2(x) = y(x, s_2) = \sum_{n=0}^{\infty} a_n(s_2)(x - x_o)^{s_2+n} \tag{2.26b}$$

are linearly independent solutions to Equation (2.1), and

$$y(x) = c_1y(x, s_1) + c_2y(x, s_2)$$

is the general solution to Equation (2.1).

d.) THE CASE WHEN $(p_0-1)^2-4q_0>0$ AND s_1-s_2 *IS* A POSITIVE INTEGER

Under this case both s_1 and s_2 are again *real*. If we set $s=s_1$ (*the larger of the two*), in Equation (2.15), then

$$n(n+s_1-s_2)a_n(s_1)=-\sum_{k=0}^{n-1}\gamma_{n,k}(s_1)a_k(s_1)$$

for $n=1,2,3,\ldots$. But $s_1-s_2=M$ (*a positive integer*) means that

$$n(n+M)\neq 0$$

or all possible integer values of n. Therefore, we may divide by this and write

$$a_n(s_1)=\frac{-\sum\limits_{k=0}^{n-1}\gamma_{n,k}(s_1)a_k(s_1)}{n(n+M)}$$

which means that we can *always solve* for all $a_n(s_1)'s$ for $n=1,2,3,\ldots$, in terms of just $a_0(s_1)$. As a result,

$$y_1(x)=y(x,s_1)=\sum_{n=0}^{\infty}a_n(s_1)(x-x_o)^{s_1+n} \tag{2.27}$$

is one solution to Equation (2.1).

Note that if we set $s=s_2$ (*the smaller of the two*), in Equation (2.15), then

$$(n+s_2-s_1)na_n(s_2)=-\sum_{k=0}^{n-1}\gamma_{n,k}(s_2)a_k(s_2)$$

for $n=1,2,3,\ldots$. But $s_1-s_2=M$ (*a positive integer*) so that

$$n(n-M)a_n(s_2)=-\sum_{k=0}^{n-1}\gamma_{n,k}(s_2)a_k(s_2), \tag{2.28}$$

for $n=1,2,3,\ldots$. When $n=M$ in this equation we get

$$M(0)a_M(s_2)=-\sum_{k=0}^{M-1}\gamma_{M,k}(s_2)a_k(s_2). \tag{2.29}$$

If the right-hand side of Equation (2.29) equals zero, then $a_M(s_2)$ is arbitrary, and then both $a_0(s_2)$ and $a_M(s_2)$ are arbitrary, which means that

$$\begin{aligned} y(x, s_2) &= \sum_{n=0}^{\infty} a_n(s_2)(x - x_o)^{s_2+n} \\ &= \sum_{n=0}^{M-1} a_n(s_2)(x - x_o)^{s_2+n} + \sum_{n=M}^{\infty} a_n(s_2)(x - x_o)^{s_2+n} \end{aligned}$$

and be written as

$$y(x, s_2) = F(x)a_0(s_2) + G(x)a_M(s_2)$$

so that $y(x, s_2)$ not only gives one solution to Equation (2.1), but it gives two linearly independent solutions $F(x)$ and $G(x)$ to Equation (2.1)! On the other hand if the right-hand side of Equation (2.29) is not equal to zero, then using $s = s_2$ (*the smaller of the two*) leads to no solution! We still, and always have $y(x, s_1)$ as one solution, however. A second solution can then be obtained using

$$y_2(x) = y_1(x) \int \left\{ \frac{e^{-\int P(x)dx}}{y_1^2(x)} \right\} dx \qquad (2.30)$$

or by the following procedure.

If we take Equation (2.13) and multiply by $(s - s_2)$, then

$$(s - s_2)L[y(x, s)] = (s - s_1)(s - s_2)^2 a_0(s)(x - x_o)^{s-2}.$$

or

$$L[(s - s_2)y(x, s)] = (s - s_1)(s - s_2)^2 a_0(s)(x - x_o)^{s-2}. \qquad (2.31)$$

If we differentiate Equation (2.31) *with respect to* s, then

$$\begin{aligned} \frac{\partial}{\partial s} L[(s - s_2)y(x, s)] &= \frac{\partial}{\partial s}\left((s - s_1)(s - s_2)^2 a_0(s)(x - x_o)^{s-2}\right) \\ &= \frac{\partial(s - s_1)}{\partial s}(s - s_2)^2 a_0(s)(x - x_o)^{s-2} \\ &\quad + (s - s_1)\frac{\partial(s - s_2)^2}{\partial s} a_0(s)(x - x_o)^{s-2} \\ &\quad + (s - s_1)(s - s_2)^2 \frac{\partial a_0(s)}{\partial s}(x - x_o)^{s-2} \end{aligned}$$

$$
\begin{aligned}
& +(s-s_1)(s-s_2)^2 a_0(s)\frac{\partial (x-x_o)^{s-2}}{\partial s} \\
= \; & (s-s_2)^2 a_0(s)(x-x_o)^{s-2} \\
& +2(s-s_1)(s-s_2)a_0(s)(x-x_o)^{s-2} \\
& +(s-s_1)(s-s_2)^2 a_0'(s)(x-x_o)^{s-2} \\
& +(s-s_1)(s-s_2)^2 a_0(s)\ln|x-x_o|(x-x_o)^{s-2}.
\end{aligned}
$$

In addition, since L operates on functions x and $\frac{\partial}{\partial s}$ operates on functions of s, (with x and s independent) we may invert the order and write,Therefore

$$
\begin{aligned}
L\left[\frac{\partial (s-s_2)y(x,s)}{\partial s}\right] = \; & (s-s_2)^2 a_0(s)(x-x_o)^{s-2} \\
& +2(s-s_1)(s-s_2)a_0(s)(x-x_o)^{s-2} \\
& +(s-s_1)(s-s_2)^2 a_0'(s)(x-x_o)^{s-2} \\
& +(s-s_1)(s-s_2)^2 a_0(s)\ln|x-x_o|(x-x_o)^{s-2}.
\end{aligned}
$$

Letting $s \to s_2$ then leads to

$$
L\left[\lim_{s\to s_2}\frac{\partial (s-s_2)y(x,s)}{\partial s}\right] = 0. \tag{2.32}
$$

This implies that

$$
y_2(x) = \lim_{s\to s_2}\frac{\partial [(s-s_2)y(x,s)]}{\partial s} \tag{2.33}
$$

is a second linearly independent solution to Equation (2.1). If we compute the result in Equation (2.33), we get

$$
y_2(x) = \lim_{s\to s_2}\frac{\partial}{\partial s}\left[\sum_{n=0}^{\infty}(s-s_2)a_n(s)(x-x_o)^{s+n}\right]
$$

$$
\begin{aligned}
= \; & \lim_{s\to s_2}\sum_{n=0}^{\infty}\left\{\frac{\partial}{\partial s}\left[(s-s_2)a_n(s)\right](x-x_o)^{s+n} + (s-s_2)a_n(s)\frac{\partial}{\partial s}(x-x_o)^{s+n}\right. \\
= \; & \lim_{s\to s_2}\sum_{n=0}^{\infty}\left\{\frac{d}{ds}\left[(s-s_2)a_n(s)\right](x-x_o)^{s+n} + \left[(s-s_2)a_n(s)\right](x-x_o)^{s+n}\ln|x-x_o|\right\}
\end{aligned}
$$

or

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} \lim_{s\to s_2} \frac{d}{ds}\left[(s-s_2)a_n(s)\right](x-x_o)^{s+n} \\ &\quad + \ln|x-x_o| \sum_{n=0}^{\infty} \lim_{s\to s_2} \left[(s-s_2)a_n(s)\right](x-x_o)^{s+n} \end{aligned}$$

so that

$$y_2(x) = \sum_{n=0}^{\infty} b_n'(s_2)(x-x_o)^{s_2+n} + \ln|x-x_o| \sum_{n=0}^{\infty} b_n(s_2)(x-x_o)^{s_2+n} \tag{2.34a}$$

where

$$b_n(s) = (s-s_2)a_n(s). \tag{2.34b}$$

To summarize our findings here, we note that cases (a) and (c) will always lead to

$$y(x) = c_1 y(x,s_1) + c_2 y(x,s_2) \tag{2.35}$$

as a general solution to Equation (2.1), while case (b) leads to

$$y(x) = c_1 y(x,s_o) + c_2 \lim_{s\to s_o} \frac{\partial y(x,s)}{\partial s} \tag{2.36a}$$

or

$$y(x) = c_1 y(x,s_o) + c_2 y(x,s_o) \int \frac{e^{-\int P(x)dx}}{y^2(x,s_o)} dx \tag{2.36b}$$

as the general solution to Equation (2.1). Finally case (d) leads to

$$y(x) = c_1 y(x,s_1) + c_2 \lim_{s\to s_2} \frac{\partial[(s-s_2)y(x,s)]}{\partial s} \tag{2.37a}$$

or

$$y(x) = c_1 y(x,s_1) + c_2 y(x,s_1) \int \frac{e^{-\int P(x)dx}}{y^2(x,s_1)} dx \tag{2.37b}$$

or

$$y_2(x) = \sum_{n=0}^{\infty} b_n'(s_2)(x-x_o)^{s_2+n} + \ln|x-x_o| \sum_{n=0}^{\infty} b_n(s_2)(x-x_o)^{s_2+n} \tag{2.37c}$$

where

$$b_n(s) = (s-s_2)a_n(s). \tag{2.37d}$$

as the general solution to Equation (2.1). In either case we can always expand the solution to Equation (2.1) about a *regular singular point*. The handout that follows these discussions shows examples of the cases we have discussed.