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Nonhomogeneous Sturm-Liouville problems and Green's functions

Consider the nonhomogeneous ODE in $a \le x \le b$:

$$\boxed{L\{w(x)\} + \Lambda\,\sigma(x)\,w(x) = \bar{L}\{w(x)\} = f(x)}\,, \tag{1a}$$

where $L(w) \equiv \left[p(x)\,w'\right]' + q(x)\,w$, with homogeneous boundary conditions (BCs):

$$w(a) + \alpha\,w'(a) = 0 \;\text{ and }\; w(b) + \beta\,w'(b) = 0 \tag{1b}$$

Recall that any problem with nonhomogeneous BCs can be transformed into one with homogeneous BCs along with a modification in the nonhomogeneous term, f, in the differential equation. (Note: Sec. 9.3 in Haberman takes $\Lambda = 0$, i.e. $L = \bar{L}$, otherwise the results are identical).

Eigenfunction Expansions

Assume

$$\boxed{w(x) = \sum_{n=0}^{\infty} C_n\,\phi_n(x)} \tag{2}$$

where the set $\{\phi_n(x)\}$ satisfies the homogeneous problem

$$L\{\phi_n(x)\} + \lambda_n\,\sigma(x)\,\phi_n(x) = 0 \tag{3}$$

with the same boundary conditions as the nonhomogeneous problem.

Recall that the expansion $g(x)$ into a series of orthogonal functions $\phi_n(x)$ with respect to the weight function $\sigma(x)$, i.e. $L(\phi_n) + \lambda_n\,\sigma(x)\,\phi_n(x) = 0$ with $L(\phi) \equiv \left[p\,\phi'\right]' + q\,\phi$,

$$g(x) = \sum_{n=0}^{\infty} A_n\,\phi_n(x) \tag{4}$$

$$A_n = \frac{\int_a^b \sigma(x_o)\,\phi_n(x_o)\,g(x_o)\,dx_o}{\int_a^b \sigma(s)\,\phi_n^2(s)\,ds} \tag{5}$$

is piecewise differentiable with respect to the operator L if $g(x)$ satisfies the boundary conditions relevant to eigenvalue problem defining the set $\{\phi_n(x)\}$. The integration variable in the numerator is chosen to be x_o to parallel Sec. 9.3 in Haberman.

Therefore, since the series (2) is differentiable we may substitute (2) into (1a)

$$\sigma(x)\sum_{n=0}^{\infty}(\Lambda-\lambda_n)C_n\,\phi_n(x)=f(x) \tag{6}$$

The coefficients C_n are determined by multiplying (6) by each member of the set $\{\phi_n(x)\}$, integrating, and using the fact that $\{\phi_n(x)\}$ is orthogonal to give

$$D_n \equiv (\Lambda-\lambda_n)C_n=\frac{\int_a^b f(x_o)\phi_n(x_o)\,dx_o}{\int_a^b \sigma(s)\phi_n^2(s)\,ds} \tag{7}$$

Thus, substituting (7) into the eigenfunction expansion (2) yields

$$w(x)=\sum_{n=1}^{\infty}\frac{D_n}{(\Lambda-\lambda_n)}\phi_n(x)=\sum_{n=1}^{\infty}\frac{1}{(\Lambda-\lambda_n)}\frac{\int_a^b f(x_o)\phi_n(x_o)\,dx_o}{\int_a^b \sigma(s)\phi_n^2(s)\,ds}\phi_n(x) \tag{8}$$

Interchanging the order of summation and integration, the solution (8) can be expressed as

$$\boxed{w(x)=\int_a^b G(x,x_o)\,f(x_o)\,dx_o} \tag{9}$$

where, provided $\Lambda\neq\lambda_n$ the *Green's function* $G(x,x_o)$ is

$$\boxed{G(x,x_o)=\sum_{n=1}^{\infty}\frac{\phi_n(x)\phi_n(x_o)}{(\Lambda-\lambda_n)\int_a^b \sigma(s)\phi_n^2(s)\,ds}=G(x_o,x)} \tag{10}$$

Example: Consider the nonhomogeneous ODE with homogeneous boundary conditions

$$u''+\Lambda u=f(x), \qquad u(0)=u(L)=0 \tag{11}$$

The corresponding homogeneous problem eigenvalues and eigenfunctions:

$$\lambda_n=\sqrt{\frac{n\pi}{L}}\,, \quad \phi_n=\sin\left(\frac{n\pi x}{L}\right) \tag{12}$$

Thus, from (10) the Green's function is

$$G(x,\xi) = \frac{2}{L}\sum_{n=0}^{\infty}\frac{1}{(\Lambda-\lambda_n)}\sin\frac{n\pi x}{L}\sin\frac{n\pi\xi}{L}$$
$$= \frac{1}{L}\sum_{n=0}^{\infty}\frac{1}{(\Lambda-\lambda_n)}\left\{\cos\left[\frac{n\pi}{L}(x-\xi)\right]-\cos\left[\frac{n\pi}{L}(x+\xi)\right]\right\} \tag{13}$$

ODE that defines $G(x,x_o)$ and its solution

If $f(x)=\delta(x-x_s)$ is substituted into (9), where the $\delta(x)$ denotes the Dirac delta function, the solution to the ODE (1) is $w(x;x_s)$ is

$$w(x;x_s) = \int_a^b G(x,x_o)\,\delta(x_o-x_s)\,dx_o = G(x,x_s) = G(x_s,x) \tag{14}$$

Recall that the Dirac delta function has the following properties:

$$\int_a^b g(x_o)\,\delta(x-x_o)\,dx_o = g(x)$$

Therefore, from (1) and (14) we conclude that the Green's function the ODE:

$$\boxed{\bar{L}\{G(x,x_o)\} = \delta(x-x_o)} \tag{15}$$

and the required homogeneous BCs (1b). The symmetry property included in (14) follows directly from the fact that $\delta(x-x_o)=\delta(x_o-x)$. We often think of the right-hand side of (15) as representing a point source.

In order to solve (15), let

$$\boxed{G(x,x_o) = \begin{cases} c_1 v_1(x), \text{ for } a \le x < x_o \\ c_2 v_2(x), \text{ for } x_o < x \le b \end{cases}} \tag{16}$$

From (15) and the fact that $\delta(x-x_o)$ vanishes except at $x=x_o$, $v_1(x)$ and $v_2(x)$ satisfy

$$\boxed{\begin{array}{ll} \bar{L}(v_1)=0 & \text{with } v_1(a)+\alpha v_1'(a)=0 \\ \bar{L}(v_2)=0 & \text{with } v_2(b)+\beta v_2'(b)=0 \end{array}} \tag{17}$$

Since each of the problems in (17) is a second-order ODE with only one BC, at this point c_1 and c_2 are undetermined. Therefore, two additional conditions are needed to uniquely specify the Green's function. One is derived from continuity of the Green's function at $x = x_o$:

$$\lim_{\varepsilon \to 0} \left[G(x, x_o + \varepsilon) - G(x, x_o - \varepsilon) \right] = 0 \tag{18}$$

and the other comes from the first integral of (15):

$$\lim_{\varepsilon \to 0} \int_{x_o - \varepsilon}^{x_o + \varepsilon} \bar{L}\left[G(x, x_o) \right] dx = 1 \tag{19}$$

Since $G(x, x_o)$ is continuous at $x = x_o$, the integral in (19) can be evaluated to give

$$\lim_{\varepsilon \to 0} \left[p(x_o + \varepsilon) G'(x, x_o + \varepsilon) - p(x_o - \varepsilon) G'(x, x_o - \varepsilon) \right] = 1 \tag{20}$$

which is the *jump condition* for the first derivative of the Green's function. Therefore, in terms of $v_1(x)$ and $v_2(x)$ with (16), the continuity (18) and jump (20) conditions can be expressed as:

$$\begin{aligned} -c_1 v_1(x_o) + c_2 v_2(x_o) &= 0 \\ -c_1 v_1'(x_o) + c_2 v_2'(x_o) &= 1/p(x_o) \end{aligned} \tag{21a,b}$$

The solution to these equation gives c_1 and c_2, which combined with (16) give:

$$\boxed{G(x, x_o) = \begin{cases} \dfrac{v_1(x) v_2(x_o)}{p(x_o) W\left[v_1(x_o) v_2(x_o) \right]}, & \text{for } a \le x < x_o \\ \dfrac{v_1(x_o) v_2(x)}{p(x_o) W\left[v_1(x_o) v_2(x_o) \right]}, & \text{for } x_o < x \le b \end{cases}} \tag{22}$$

where W denotes the Wronskian.

Example: Consider the nonhomogeneous ODE with homogeneous boundary conditions

$$u'' - u = f(x), \qquad u(0) = u(L) = 0 \tag{23}$$

For this problem it is straightforward to see that $v_1(x) = \sinh(x)$ and $v_2(x) = \sinh(L - x)$, and

$$G(x, x_o) = \frac{-1}{\sinh(L)} \begin{cases} \sinh(x) \sinh(L - x_o), & \text{for } 0 \le x < x_o \\ \sinh(L - x) \sinh(x_o), & \text{for } x_o < x \le L \end{cases} \tag{24}$$