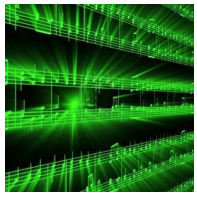
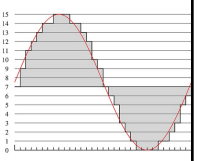




Lecture #4 – Nyquist-Shannon Sampling Theorem

ESE 1500 – Digital Audio Basics

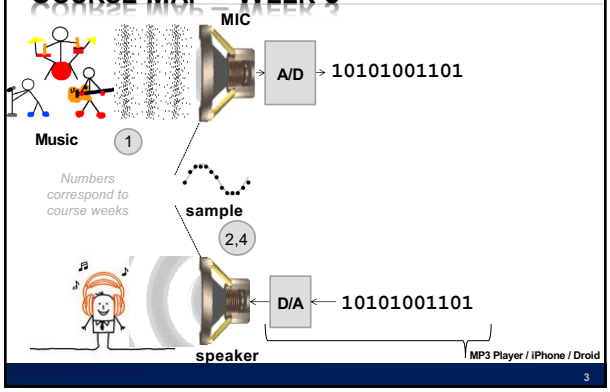
1

Lecture Topics

- **Part 1:**
  - Where are we on course map?
  - Sampling/Quantization Review
  - **Impact of Sampling Rates**
  - **Aliasing**
- **Interlude: Visual Aliasing**
- **Part 2:**
  - **Aliasing Math**
  - **Nyquist-Shannon Sampling Rate**
  - References

2

Course Map – Week 3



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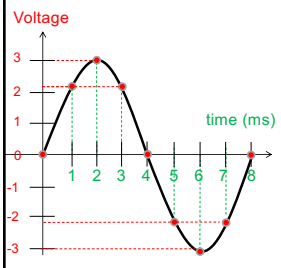
Sampling vs Quantization Review

4

ADC – Sampling

- ***Sampling*: breaking independent variable (time) into intervals**
- ***Quantization*: breaking dependent variable (voltage) into levels**



Samples @ 1ms intervals:	Quantized into 7 levels	Levels digitized into 3-bits
{ 0 ms, 0 Volts }	{ 0 ms, 0 Volts }	011
{ 1 ms, 2.2 Volts }	{ 1 ms, 2 Volts }	101
{ 2 ms, 3 Volts }	{ 2 ms, 3 Volts }	110
{ 3 ms, 2.2 Volts }	{ 3 ms, 2 Volts }	101
{ 4 ms, 0 Volts }	{ 4 ms, 0 Volts }	011
{ 5 ms, -2.2 Volts }	{ 5 ms, -2 Volts }	001
{ 6 ms, -3 Volts }	{ 6 ms, -3 Volts }	000
{ 7 ms, -2.2 Volts }	{ 7 ms, -2 Volts }	001
{ 8 ms, 0 Volts }	{ 8 ms, 0 Volts }	011

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Two Knobs

1. **Quantization level (bits/sample)**
2. **Sampling rate (samples/second)**

- **Impact Quality of sound**
  - Potential error introduced in reconstruction → noise
- **Impact costs (resources -- #bits needs to store)**

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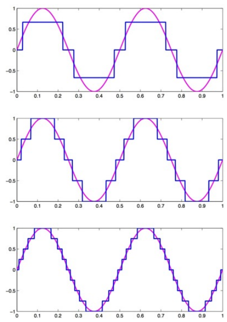
Effect of Increasing Quantization



- **Dividing dependent variable up into more levels**
  - Increasing resolution at each sample
  - Doesn't change the # of samples itself!

7

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Effect of Increasing Sampling Rate

- **Increasing how often we take samples also helps**
  - Much like quantization…
    - 1 bit was too few, 16 bits was more than enough
    - Is there a sweet spot for the sampling rate?
      - Focus for this week.

8

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Both (Quantization, Sampling) Impact Storage

- **How many bytes for a 3 minute song sampled at 8b precision and 1000 samples/s?**
- **at 2000 samples/s?**
- **16b precision at 2000 samples/s?**

9

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Key Question

- **What sampling rate should we use?**

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Definition of Good Sampling

- **Definition of proper sampling:**
  - Let's say you've sampled an analog signal…
  - If you can **exactly** reconstruct the analog signal from the samples
    - You have done the sampling properly!
  - Essentially: if you can reverse the process…
    - You've capture enough information about the signal
- **Can we formalize this a bit more?**
  - Yes, next few slides will try….

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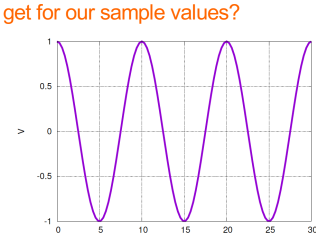
Sample at Frequency

- **Preclass 1: What happens if we sample 100Hz signal at 100Hz?**
  - What do we get for our sample values?



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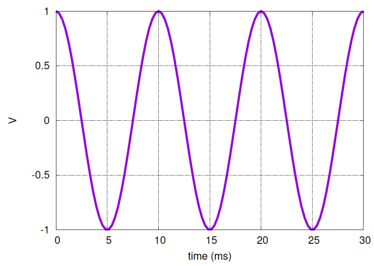
Sampling – What is the minimum?



- Sampling at frequency doesn't work.

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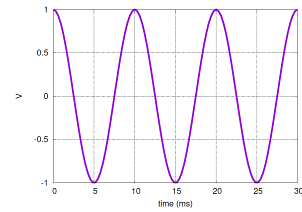
Sampling – What is the minimum?



- **How much do we need to capture to reconstruct it?**
  - If we sample at 200 Hz, capture peaks & troughs of signal
  - Sample rate: 2 x frequency = 200 Hz

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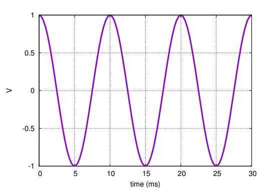
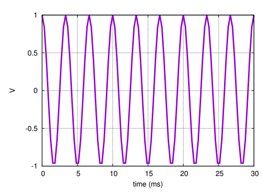
Preclass 2 – Sample at 200Hz





	0 ms	5 ms	10 ms	15 ms	20 ms	25 ms	36 ms	Freq.
left								
right								

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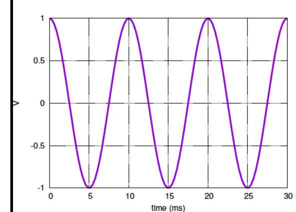
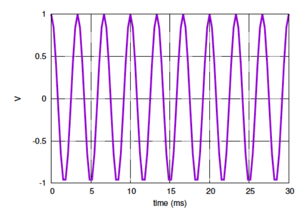
200 Hz Sample





- **What happened here?**
  - What did we get for the two cases?
  - Why?

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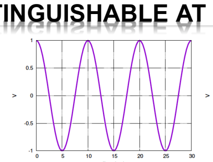
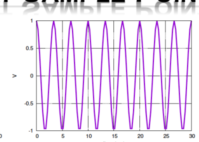
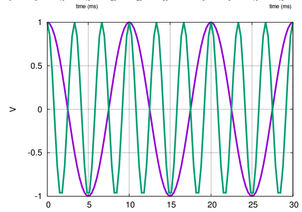
Indistinguishable at Sample Points







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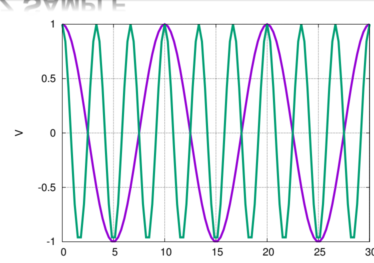
200 Hz Sample



- **Sampling *below* frequency doesn't work**
- **Cannot let signal "wiggle" around between samples**
  - Change direction
- **Sample too infrequently, can miss signal behavior**

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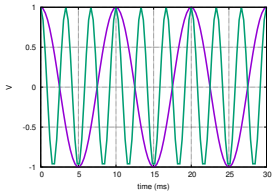
Initial Observation

- **Observe:** we must, at least, sample at twice the frequency of the signal we are trying to capture
  - If sample at a lower frequency, signal may change directions between samples
  - This gives us a **lower bound** on sample rate



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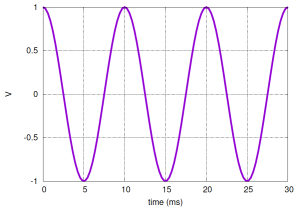
Sampling – What is the minimum?



- **Observation:**
  - If we sample at 200 Hz, capture peaks & troughs of signal
  - Sample rate: 2 x frequency = 200 Hz
  - Must sample at 2x frequency so doesn't wiggle/change-direction between samples

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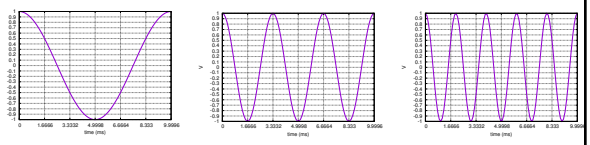
Preclass 3 – Sample 600 Hz



	0 ms	1.66 ms	3.32 ms	4.99 ms	6.66 ms	8.33 ms	9.99 ms	freq
left								
middle								
right								

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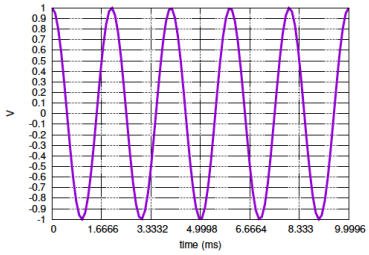
Preclass 3 – 500Hz



- **Is this properly sampled?**
- **What did we get?**
- **How does sample rate relate to frequency?**

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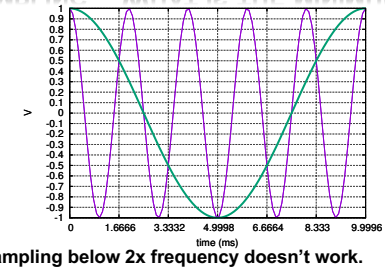
Sampling – What is the minimum?



- **Sampling below 2x frequency doesn't work.**
- **Also leads to reconstruction error**
  - We not only lose information…
    - …but when we 'reconstruct' the signal from the samples alone…
    - We will reconstruct at a lower frequency!
    - This phenomenon is called: aliasing

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Interlude

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Video

- How many frames/second for video (TV, Film?)
- http://www.youtube.com/watch?v=jHS9JGkEOmA

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Aliasing in Movies

- **Called visual aliasing**
  - See it all the time on TV/Film
    - Wheels tend to move backwards on moving cars…why?
  - What is it?
    - Primer: Movies are just pictures (frames) flying by quickly
    - Movies "sample" real life at roughly 24 frames per second
  - What did we just see?
    - When changes occur faster than ½ f_s , may get aliasing.
    - Film Example:
      - If **light to dark transitions** occur faster than ½ f_s aka: 12 frame/sec
      - Aliasing will occur…

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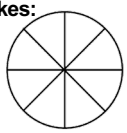
The "Wagon Wheel" Effect

- **Consider a wagon wheel with 8 spokes:**



  - Let's say it turns at a rate of 3 revolutions per second clockwise
    - That's 180 rpm
  - On film this wheel will appear to stand still! Why?

$$\frac{\left(3\ \frac{revolutions}{sec}\right)\times\left(8\ \frac{spokes}{revolution}\right)}{\left(24\ \frac{frames}{sec}\right)}=1\ \frac{spoke}{frame}$$

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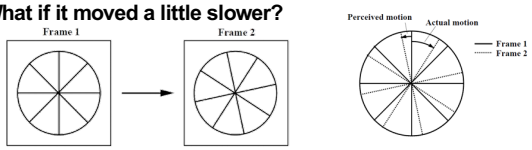
The "Wagon Wheel" Effect

- **What if it moved a little slower?**



  - Let's say it turns at a rate of 2.5 revolutions per second clockwise

$$\frac{\left(2.5\ \frac{revolutions}{sec}\right)\times\left(8\ \frac{spokes}{revolution}\right)}{\left(24\ \frac{frames}{sec}\right)}=.83\ \frac{spoke}{frame}$$

  - Our brain could interpret this in two possible ways:
    - Wheel has moved clockwise by 83% of spoke interval in clockwise direction
    - OR: wheel has moved counter-clockwise by 17%
      - Our brains prefer this view! So we see the wheel moving backwards! *(thanks aliasing!)*

Fool your brain: http://www.youtube.com/watch?v=jHS9JGkEOmA

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Part 2

Aliasing Math
Nyquist-Shannon Sampling

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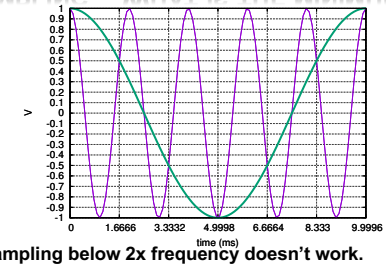
Sampling – What is the minimum?



- **Sampling below 2x frequency doesn't work.**
- **Also leads to reconstruction error**
  - We not only lose information…
    - …but when we 'reconstruct' the signal from the samples alone…
    - We will reconstruct at a lower frequency!
    - This phenomenon is called: aliasing

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Aliasing Mathematical Derivation

- **500Hz cosine:** $\cos(2\pi \cdot 500 \cdot t)$
- **Sampled at 600Hz**
  - Only look at t=I/600
  - I is the index for samples
- **So, our discrete version:** $\cos\left(2\pi \cdot 500 \cdot \left(\frac{I}{600}\right)\right)$
- **Simplify :** $\cos\left(2\pi \cdot \left(\frac{5}{6}\right) \cdot I\right)$
- **Rearrange :** $\cos\left(2\pi \cdot I - 2\pi \cdot \left(\frac{1}{6}\right) \cdot I\right)$

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Mathematical Manipulation

- **500Hz cosine:** $\cos(2\pi \cdot 500 \cdot t)$
- **Sampled at 600Hz**
- **Now :** $\cos\left(2\pi \cdot I - 2\pi \cdot \left(\frac{1}{6}\right) \cdot I\right)$
  - I is an integer.
  - $\cos(x + 2\pi) = \cos(x)$
- **Apply:** $\cos\left(-2\pi \cdot \left(\frac{1}{6}\right) \cdot I\right)$
  - $\cos(-x) = \cos(x)$
- **Apply:** $\cos\left(2\pi \cdot \left(\frac{1}{6}\right) \cdot I\right)$

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Aliasing Derivation

- **500Hz cosine:** $\cos(2\pi \cdot 500 \cdot t)$
- **Sampled at 600Hz**
- **discrete version:** $\cos\left(2\pi \cdot 500 \cdot \left(\frac{I}{600}\right)\right)$
- **Simplified to:** $\cos\left(2\pi \cdot \left(\frac{1}{6}\right) \cdot I\right)$
- **Same as:** $\cos\left(2\pi \cdot 100 \cdot \left(\frac{I}{600}\right)\right)$
  - Which would correspond to 100Hz signal!

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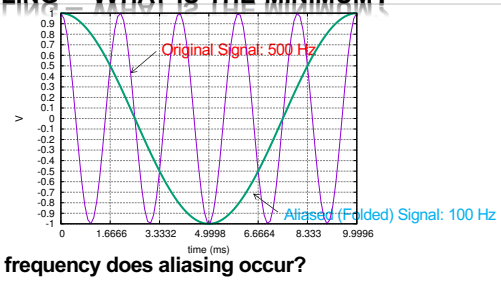
Sampling – What is the minimum?



- **What frequency does aliasing occur?**
  - Original Signal's Frequency: 500 Hz
    - Sampling Rate: **600 Hz**
  - Aliasing occurs at: **600 Hz** – 500 Hz = 100 Hz
    - Also referred to as "Folding" – signal has "folds over" as if it were lower frequency

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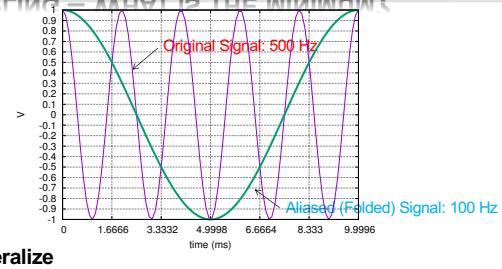
Sampling – What is the minimum?



- **Generalize**
  - F' = frequency mod SampleRate (subtract out integer 2π terms)
  - Alias frequency is
    - F' if F'<SampleRate/2
    - SampleRate-F' if SampleRate/2 < F' < SampleRate

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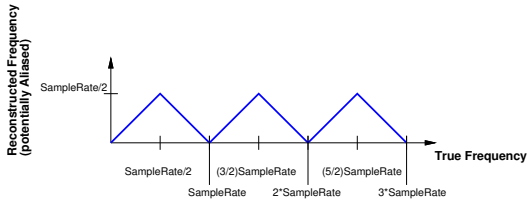
Sampling – What is the minimum?

- **Generalize**
  - F' = frequency mod SampleRate (subtract out integer 2π terms)
  - Alias frequency is
    - F' if F'<SampleRate/2
    - SampleRate-F' if SampleRate/2 < F' < SampleRate



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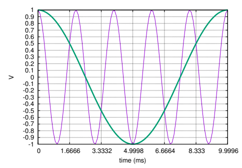
Next Observation

- **Observation:** sampling at less than twice the frequency of the signal can lead to **aliasing**
  - Reinforces will need to sample at, at least, twice the frequency of our sample



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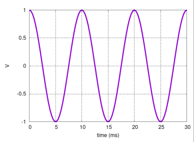
Sampling Rate

- **Established (by counterexamples) that we can sample too infrequently**
  - **Necessary** to sample at 2x highest frequency present
- **Haven't shown clearly that 2x is sufficient**
  - (won't in this class)
  - Just giving you intuition
    - Capture all the peaks and troughs
    - Sufficient to guarantee signal doesn't "wiggle" between samples



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Sampling – What is the minimum?

- **Harry Nyquist**
  - Electronic Engineer for AT&T from 1917 to 1954
  - Published paper in 1928 defining the: Sampling Theorem
    - **Nyquist Sampling Rate** = 2 x frequency of signal
      - Anything less: *under-sampling* – leads to aliasing
      - Anything more: *over-sampling* – waste of space?

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Possibly Open Questions (so far)

- **Do all signals have frequency?**
  - Is this issue of 2x frequency of signal well defined?
- **Are pure tones an adequate model of frequencies in signals?**
- **Will be solidifying this in 2 weeks.**

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Big Ideas

- **Sample at twice the maximum frequency**
  - Can reconstruct perfectly
- **If have frequencies > SampleRate/2**
  - Will get aliasing … as high frequencies fold

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Learn More

- **ESE224 – Signal Processing**
- **ESE531 – Digital Signal Processing**

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ADMIN

- **Remember feedback**
- **Lab Writeup Due Monday**
- **Next Lab on Monday**
  - Preclass (bit longer this time)
  - Lab out …
- **Office Hours**
  - DeHon: W4:30pm-5:30pm
  - Peter: TODO

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REFERENCES

- S. Smith, "The Scientists and Engineer's Guide to Digital Signal Processing," 1997.
- http://en.wikipedia.org/wiki/Nyquist_frequency
- http://en.wikipedia.org/wiki/Nyquist_rate
- http://en.wikipedia.org/wiki/Oversampling
- http://en.wikipedia.org/wiki/Sampling_rate
- http://en.wikipedia.org/wiki/Hearing_range
- http://electronics.howstuffworks.com/telephone6.htm
- B. Olshausen, "Aliasing", PSC 129 – Sensory Processes Course Notes, UC Davis

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