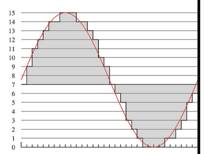


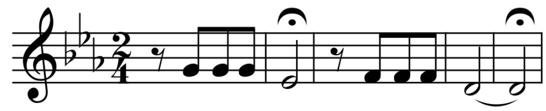
Lecture #07– Frequency Domain

ESE 1500 – Digital Audio Basics

1

Teaser

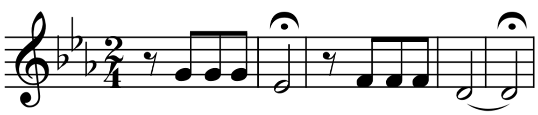
- Play this on piano:



2

Teaser

- Play



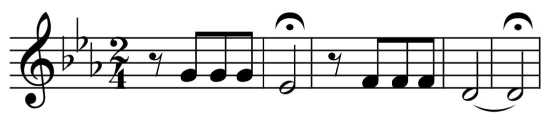
3 eighth notes   1 half note

Cheat: G4 E4^b F4 D4

3

Information

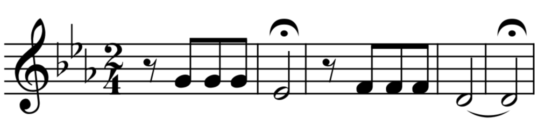
- 1s / quarter note ➔ 10s of sound
- How many bits to represent 10s of sound with 16b samples and 44KHz sampling?
  - 44K Hz x 16b/sample x 10s = 7040K =7Mbits



4

Representation

- How does musical staff represent sound?
  - What does vertical position represent?
  - Note shape?



5

Musical Staff – Height = Frequency

- Source: http://www.rochars.org/recorder_frequency_ranges_big

6

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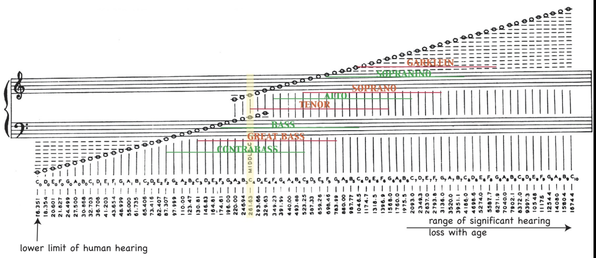
Musical Staff – Height = Frequency



- Source: http://www.rochars.org/recorder_frequency_ranges_big

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Musical Notation – Shape = Time

Whole Note	4 Counts	Whole Rest	4 Counts
Half Note	2 Counts	Half Rest	2 Counts
Quarter Note	1 Count	Quarter Rest	1 Count
Eighth Note	½ Count	Eighth Rest	½ Count
1/16th Note	¼ Count	1/16th Rest	¼ Count

- Source: https://www.pikpng.com/downpngs/hiJiwww_note-values-in-4-4-time-music-notes/

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8

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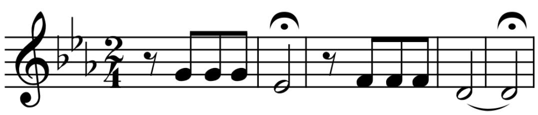
Frequency Representation

- **There are other ways to represent**
  - Frequency representation particularly efficient



392 311 348 294

Frequencies in Hertz

9

9

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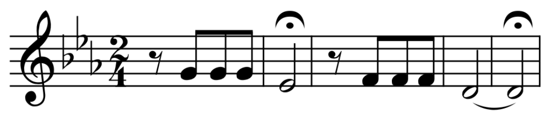
Frequency Representation

- **How much information is this musical staff communicating?**
- **How many keys on piano? → bits/note**



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Hamburg Steinway D-274 Piano photo by Karl Kunde

https://commons.wikimedia.org/wiki/File:D274.jpg

11

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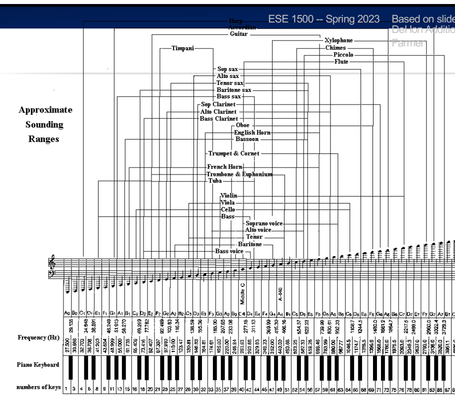


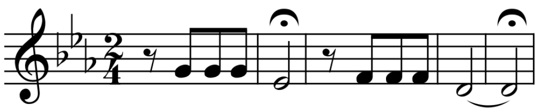
Larry Solomn: http://solomonsmusic.net/insrange.htm

12

12

Frequency Representation

- How much information is this musical staff communicating?
- How many keys on piano? → bits/note (or rest)
- Let's say 8b duration
- How many bits for 10 notes, rests?
  - (7b/note+8b/duration) x 10 note = 150 bits?



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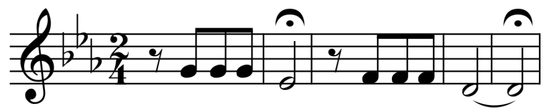
Conclude

- Can represent common sounds much more compactly in frequency domain than in time-sample domain
  - Frequency domain ~ 150b
  - Time-sample domain ~ 7Mb



14

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Lecture Topics

- Part 1
  - Teaser: frequency representation
  - Where are we on course map?
  - Frequency Domain
- Part 2: Vector Background
- Part 3: The Fourier Series
  - can represent any signal in frequency domain
- References

15

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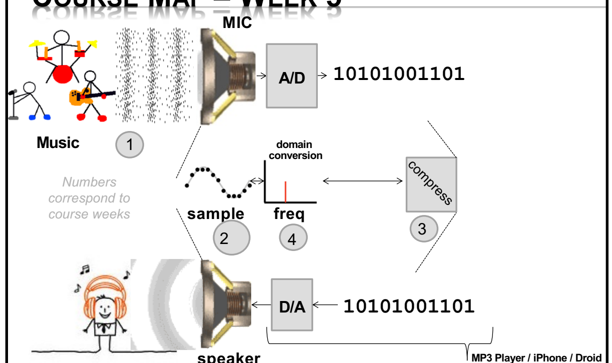
Course Map – Week 5



16

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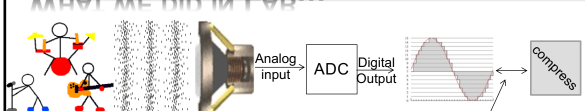
What we did in Lab…



- Week 1: Converted Sound to analog voltage signal
  - a "pressure wave" that changes air molecules w/ respect to time
  - a "voltage wave" that changes amplitude w/ respect to time
- Week 2: Sampled voltage, then quantized it to digital sig.
  - Sample: Break up independent variable, take discrete 'samples'
  - Quantize: Break up dependent variable into n-levels (need 2^n bits to digitize)
- Week 3: Compress digital signal
  - Use even less bits without using sound quality!
- Week 4 (upcoming): Before we compress…
  - Put our 'digital' data into another form…BEFORE we compress…less stuff to compress!

17

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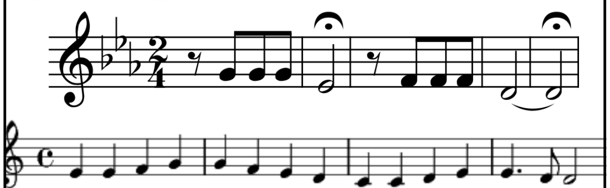
Background

What is the Frequency Domain?

18

Musical Representation

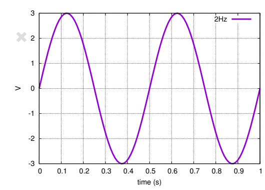


- With this compact notation
  - Could communicate a sound to pianist
  - Much more compact than 44KHz time-sample amplitudes (fewer bits to represent)
  - Represent frequencies

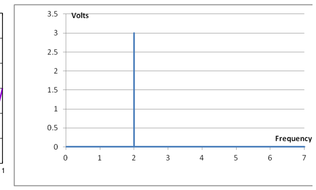
19

Time-Domain & Frequency-Domain

- As an example…let's say we have a pure tone
  - If period: T = 1/2 and Amplitude = 3 Volts
  - $s(t) = A\,sin(2\pi f t) = 3\,sin(2\pi 2 t)$



Time domain representation

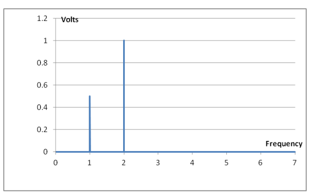


Frequency domain representation

20

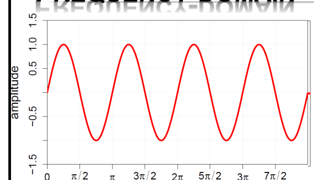
Frequency-Domain

- Of course, not all signals are this simple
- For example: $s(t) = \sin(2\times 2\pi\times t) + \frac{1}{2}\sin(2\pi\times t)$
- Question: What will the frequency representation look like?

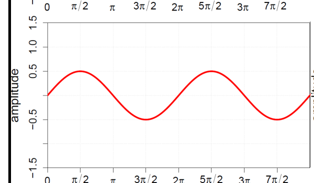


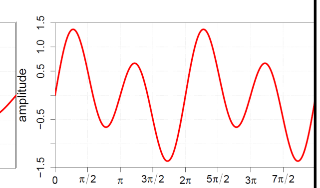
21

Frequency-Domain



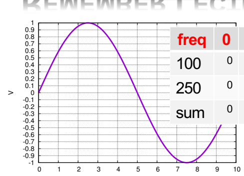
- How about the time-domain?
  - Plot $\sin(2\times 2\pi\times t)$
  - Plot $\frac{1}{2}\sin(2\pi\times t)$
  - Sum: $\sin(2\times 2\pi\times t) + \frac{1}{2}\sin(2\pi\times t)$
  - ***Notice how it was easier to plot the frequency domain representation***





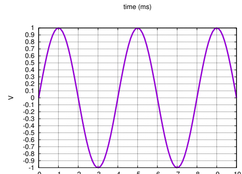
22

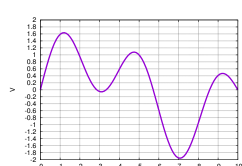
Remember Lecture 5: Preclass 1



freq	0	1	2	3	4	5	6	7	8	9	10
100	0	0.6	0.95	0.95	0.6	0	-0.6	-0.95	-0.96	-0.6	0
250	0	1.0	0	-1	0	1	0	-1	0	1	0
sum	0	1.6	0.95	-0.05	0.6	1	-0.6	-1/95	-0.95	0.4	0

$\sin(100\times 2\pi\times t) + \sin(250\times 2\pi\times t)$

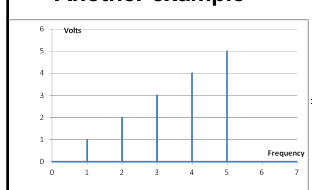


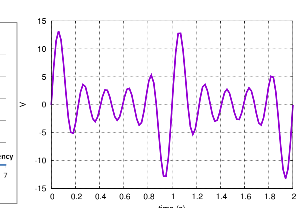


23

Frequency-Domain

- Another example





The time domain plot on the right is really the sum of 5 sinusoids, where 5 Hz is the strongest component of the signal

24

FREQUENCY-DOMAIN

- **So far…**
  - we have seen how a signal written as:
    - a sum of sines of different frequencies
  - can have a ***frequency domain representation***
- **Each sine component…**
  - is more or less important depending on its **coefficient**
  - Example: $s(t) = 1\sin(2\times 2\pi\times t) + \frac{1}{2}\sin(2\pi\times t)$
- **Can any arbitrary signal be represented as a sum of sines?**
  - No. But the idea has potential, let's explore it!

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Part 2

VECTOR BACKGROUND

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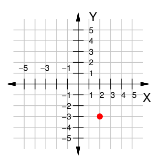
VECTOR SPACE

- **We're familiar with multi-dimensional spaces and vector representation**
  - E.g. Cartesian Coordinates in 2 Space
    - 2 dimensions X, Y
    - Represent points as vector with 2 elements (x,y)
  - Preclass 4a
    - What is the (x,y) coordinate of the red dot?



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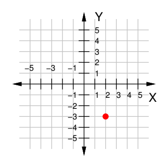
VECTOR SPACE

- **We're familiar with multi-dimensional spaces and vector representation**
  - E.g. Cartesian Coordinates in 2 Space
    - 2 dimensions X, Y
    - Represent points as vector with 2 elements (x,y)
  - Can easily extend to 3 Space
    - (x,y,z)
  - Harder to visualize, but could extend to any number of dimensions
    - (d1,d2,d3,d4,d5,….)



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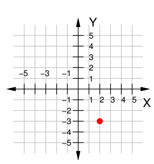
ORTHOGONAL

- **Perpendicular**
- **Point in different directions**
- **X and Y axes are orthogonal to each other**



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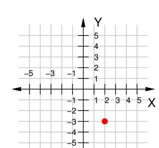
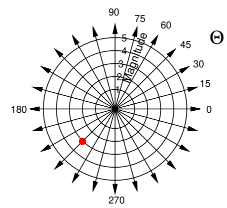
ORTHOGONAL BASIS

- **We can describe any point in the space by a linear combination of orthogonal basis elements**
  - E.g. Cartesian Coordinates in 2 Space
    - x -- [1,0]
    - y -- [0,1]
    - Any point:
      - a*x + b*y = [a,b]
  - Orthogonal – no linear scaling of one gives the other
    - Dot products are zero
      - $X \cdot Y = [0,1] \cdot [1,0] = 1\times 0 + 0\times 1 = 0$
- **Combine by linear superposition**



Different Representations

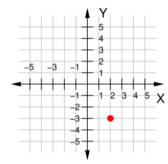
- **We can also represent points in 2-space in polar coordinates**
  - A different orthogonal basis
    - (magnitude, Θ)
- **What is the polar coordinate of the red dot? (4b)**

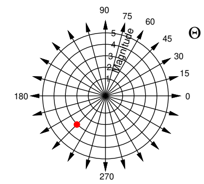


31

Can Change Representations

- **Both Cartesian and Polar Coordinates can describe points in the same space.**
  - How do we change polar to Cartesian? (4c)
  - What is the Cartesian coordinate for the red dot? (4d)





32

Complex Numbers

- **Complex Numbers are an example of this**
  - Real dimension
  - Imaginary dimension
- **Cartesian version:** a+bi
- **Polar (Magnitude, angle) version:** $M \times e^{i\theta}$
- Euler's Formula: $e^{i\theta} = \cos\theta + i\sin\theta$

33

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Part 3:

The Fourier Series

34

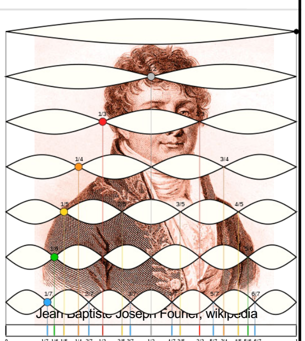
History…

- **Fourier series:**
  - Any ***periodic*** signal can be represented as a sum of simple periodic functions: *sin* and *cos*

sin(nt) and cos(nt)

where n = 1, 2, 3, …

These are called the ***harmonics*** of the signal



Jean Baptiste Joseph Fourier, wikipedia

35

Fourier Series – More Formally

The Fourier Theorem states that any *periodic* function f(t) of period L can be cast in the form:

$$f(t) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi t}{L} + b_n \sin\frac{n\pi t}{L}\right)$$

The constants: a_0, a_n, and b_n are called the Fourier coefficients of *f(t)*

[also a complex number version that uses complex coefficient and $e^{i\theta}$ instead of cos/sin]

36

Fourier Series – More Formally

The Fourier Theorem states that any *periodic* function f(t) of period L can be cast in the form:

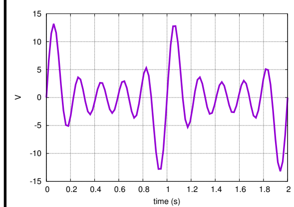
$$f(t) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi t}{L} + b_n \sin\frac{n\pi t}{L}\right)$$

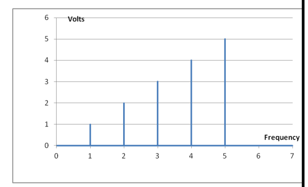
The constants: a_0, a_n , and b_n are called the Fourier coefficients of *f(t)*

The constants: a_0, a_n , and b_n become our **vector** describing a signal in frequency space (domain)

Frequency-Domain

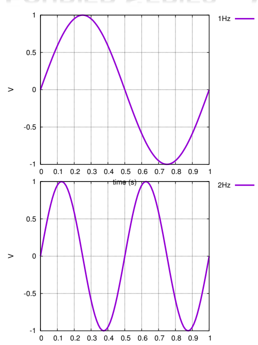
$$f(t) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi t}{L} + b_n \sin\frac{n\pi t}{L}\right)$$

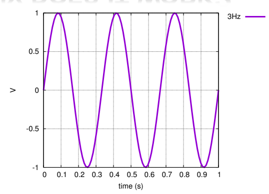




Frequency Representation = [0,0,1,0,2,0,3,0,4,0,5,0,0,0,….]

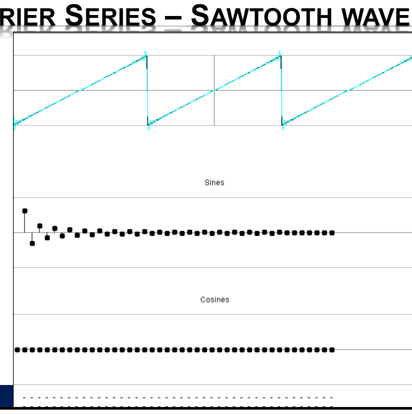
Fourier Series – Why does it work?





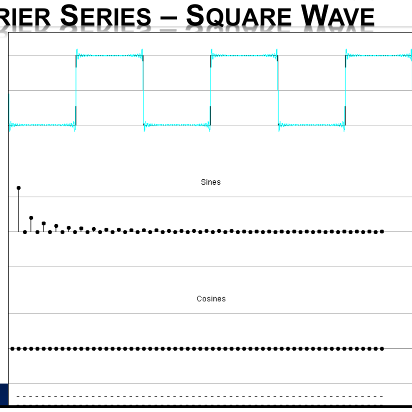
- The cos(*nx*) and sin(*nx*) functions form an orthogonal basis:
  - allow us to represent any periodic signal by taking a *linear combination*
  - of the basis functions without interfering with one another
  - AKA: superposition works!

Fourier Series – Sawtooth Wave



(falstad.com/fourie

Fourier Series – Square Wave



(falstad.com/fourie

Fourier Series (Review of Key Points)

- **The idea of the series:**
  - Any ***PERIODIC*** wave can be represented as simple sum of sine and cosine waves
- **2 Caveats:**
  - Linearity:
    - The series only holds while the system it is describing is *linear* because it relies on the superposition principle
    - -aka – adding up all the sine waves is superposition in action
  - Periodicity:
    - The series only holds if the waves it is describing are periodic
    - Non-periodic waves are dealt with by the Fourier Transform
      - We will examine that in Lecture 9 (next Monday)

Nyquist

- **Remember we said we needed to sample at twice the maximum frequency**
  - Now see all signals can be represented as a linear sum of frequencies
  - …and the frequency components are orthogonal
    - Can be extracted and treated independently

43

Big Ideas

- **Can represent signals in frequency domain**
  - Different basis – basis vectors of sines and cosines
- **Often more convenient and efficient than time domain**
  - Remember musical staff

$$f(t) = \frac{a_o}{2} + \sum_{n=1}^{N} [a_n \cos(nt) + b_n \sin(nt)]$$

44

Learn More

- **ESE3250 – whole course on Fourier Analysis**
- **ESE2240 – signal processing**
- **ESE2150, 3190, 4190 – reason about behavior of circuits in time and frequency domains**

45

Reminder

- **Feedback**
- **Lab 2 due today**
- **Lab 3 today**

46

References

- **S. Smith, "The Scientists and Engineer's Guide to Digital Signal Processing," 1997.**
- **https://betterexplained.com/articles/an-interactive-guide-to-the-fourier-transform/**

47