Discrete signals

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Discrete signals

- Discrete and finite time index $n = 0, 1, \ldots, N-1 = [0, N-1]$.
- Discrete signal x is a function mapping $[0, N-1]$ to a real value $x(n)$

$$x : [0, N-1] \to \mathbb{R}$$

- The values that the signal takes at time index n is $x(n)$
- Sometimes it will make sense to talk about complex signals

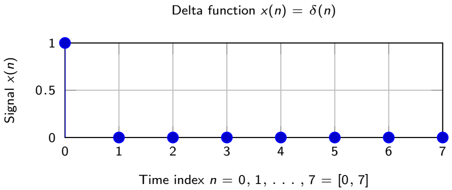
$$x : [0, N-1] \to \mathbb{C}$$

- The values $x(t) = x_R(t) + j\, x_I(t)$ are complex numbers
- Space of signals = space of N-dimensional vectors $\mathbb{R}^N$ or $\mathbb{C}^N$

Deltas (impulses, spikes)

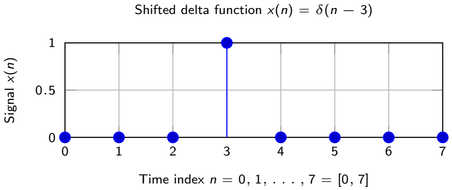
- The discrete delta function $\delta(n)$ is a spike at (initial) time $n = 0$

$$\delta(n) = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{else} \end{cases}$$



- The shifted delta function $\delta(n - n_0)$ has a spike at time $n = n_0$

$$\delta(n-n_0) = \begin{cases} 1 & \text{if } n = n_0 \\ 0 & \text{else} \end{cases}$$

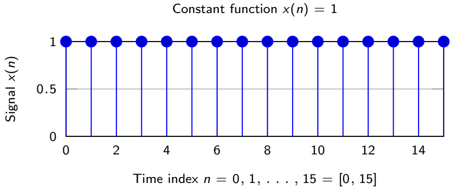


- This is not a new definition, just a time shift

Constants and square pulses

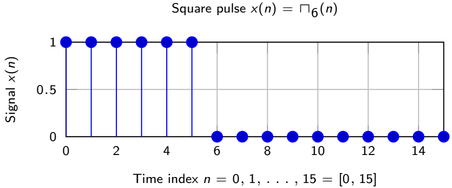
- A constant function $x(n)$ has the same value c for all n

$$x(n) = c, \quad \text{for all } n$$



- A square pulse of width M, $\sqcap_M(n)$, equals one for the first M values

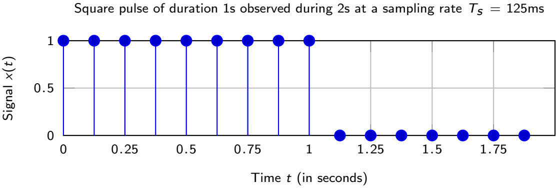
$$\sqcap_M(n) = \begin{cases} 1 & \text{if } 0 \leq n < M \\ 0 & \text{if } M \leq n \end{cases}$$



- Can consider shifted pulses $\sqcap_M(n - n_0)$, with $n_0 < N - M$

Units: Sampling time and signal duration

- Sampling time T_s $\Rightarrow$ Time elapsed between indexes n and $n+1$
  $\Rightarrow$ Sampling frequency $f_s := 1/T_s$
- Time index n represents actual time $t = nT_s$

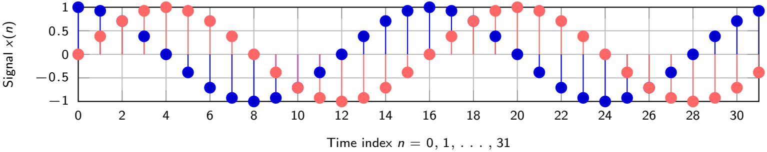


- Signal duration $T = NT_s$ $\Rightarrow$ Time length of signal
  $\Rightarrow$ The last sample is "held" during T_s time units

Discrete cosines and sines

- For a signal of duration N define (assume N is even):
  - $\Rightarrow$ Discrete cosine of discrete frequency k $\Rightarrow x(n) = \cos(2\pi kn/N)$
  - $\Rightarrow$ Discrete sine of discrete frequency k $\Rightarrow x(n) = \sin(2\pi kn/N)$

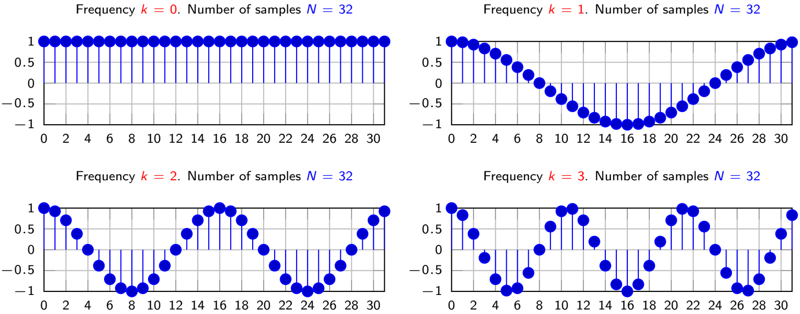
Cosine $x(n) = \cos(2\pi kn/N)$ and sine $x(n) = \sin(2\pi kn/N)$. Frequency $k = 2$ and number of samples $N = 32$.



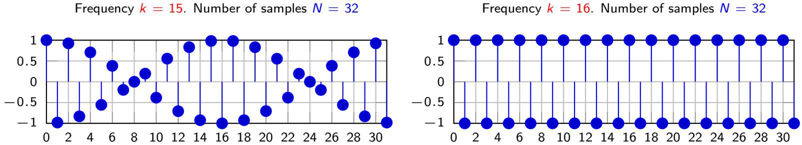
- Frequency k is discrete. I.e., $k = 0, 1, 2, \ldots$
  - $\Rightarrow$ Have an integer number of complete oscillations

Cosines of different frequencies (1 of 2)

- Discrete frequency $k = 0$ is a constant
- Discrete frequency $k = 1$ is a complete oscillation
- Frequency $k = 2$ is two oscillations, for $k = 3$ three oscillations ...

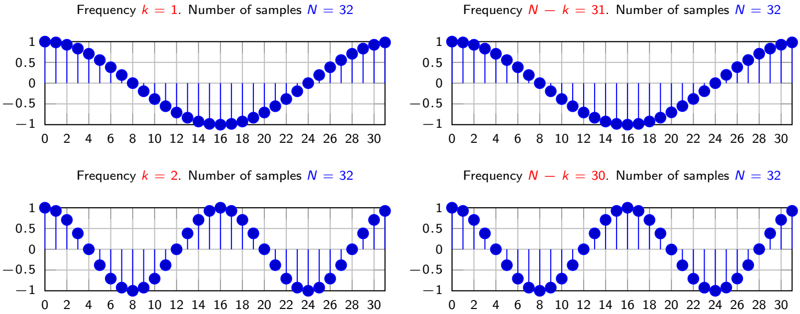


- Frequency k represents k complete oscillations
- Although for large k the oscillations may be difficult to see



- Do note that we can't have more than $N/2$ oscillations
  - $\Rightarrow$ Indeed $1 \rightarrow -1 \rightarrow 1, \rightarrow -1, \ldots$
  - $\Rightarrow$ Frequency $N/2$ is the last one with physical meaning
- Larger frequencies replicate frequencies between $k = 0$ and $k = N/2$

- Frequencies k and $N - k$ represent the same cosine



- Actually, if $k + l = N$, cosines of frequencies k and l are equivalent
- Not true for sines, but almost. The signals have opposite signs

Discrete frequencies and actual frequencies

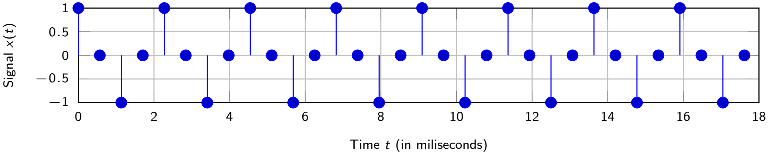
- What is the discrete frequency k of a cosine of frequency f_0?
- Depends on sampling time T_s, frequency $f_s = \frac{1}{T_s}$, duration $T = NT_s$
- Period of discrete cosine of frequency k is T/k (k oscillations)
- Thus, regular frequency of said cosine is $\Rightarrow f_0 = \dfrac{k}{T} = \dfrac{k}{NT_s} = \dfrac{k}{N} f_s$
- A cosine of frequency f_0 has discrete frequency $k = (f_0/f_s)N$
- Only frequencies up to $N/2 \leftrightarrow f_s/2$ have physical meaning
- Sampling frequency f_s $\Rightarrow$ Cosines up to frequency $f_0 = f_s/2$

Use of units example

- Generate $N = 32$ samples of an A note with sampling frequency $f_s = 1,760$Hz
- The frequency of an A note is $f_0 = 440$Hz. This entails a discrete frequency

$$k = \frac{f_0}{f_s}N = \frac{440\text{Hz}}{1,760\text{Hz}}32 = 8$$

The A note observed during $T = NT_s = 18.2$ms with a sampling rate $f_s = 1,760$Hz



- Alternatively $\Rightarrow x(n) = \cos\left[2\pi kn/N\right] = \cos\left[2\pi(f_0/f_s)Nn/N\right]$
- Which simplifies to $\Rightarrow x(n) = \cos\left[2\pi(f_0/f_s)n\right] = \cos\left[2\pi f_0(nT_s)\right]$

Noninteger frequencies

- The frequency k need not be integer but it's not a discrete cosine
  $\Rightarrow$ Sampled cosine $\Rightarrow x(n) = \cos(2\pi k n/N)$
  $\Rightarrow$ Sampled sine $\Rightarrow x(n) = \sin(2\pi k n/N)$

- Discrete sine and cosine have complete oscillations
- Sampled sine and cosine may have incomplete oscillations

- Discrete sine and cosine are used to define Fourier transforms (later)

Inner products and energy

Inner product

- Given two signals x and y define the inner product of x and y as

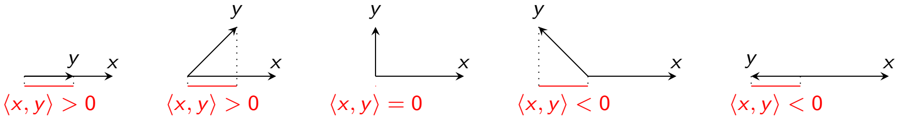
$$\begin{aligned}\langle x, y\rangle &:= \sum_{n=0}^{N-1} x(n) y^*(n) \\ &= \sum_{n=0}^{N-1} x_R(n) y_R(n) + \sum_{n=0}^{N-1} x_I(n) y_I(n) + j \sum_{n=0}^{N-1} x_I(n) y_R(n) - j \sum_{n=0}^{N-1} x_R(n) y_I(n)\end{aligned}$$

- Inner product between vectors x and y, just with different notation

- Inner product is a linear operations $\Rightarrow \langle x, y+z\rangle = \langle x, y\rangle + \langle x, z\rangle$
- Reversing order equals conjugation $\Rightarrow \langle y, x\rangle = \langle x, y\rangle^*$

Inner product interpretation

- Signal inner product has same intuition as vector inner product
  - $\Rightarrow$ Inner product $\langle x, y \rangle$ is the projection of y on x
  - $\Rightarrow$ The value of $\langle x, y \rangle$ is how much of y falls in x direction
- E.g., if $\langle x, y \rangle = 0$ the signals are orthogonal. They are "unrelated"



Norm and energy

- Following the algebra analogies, define the norm of signal x as

$$\|x\| := \left[\sum_{n=0}^{N-1} |x(n)|^2\right]^{1/2} = \left[\sum_{n=0}^{N-1} |x_R(n)|^2 + \sum_{n=0}^{N-1} |x_I(n)|^2\right]^{1/2}$$

- More important, define the energy of the signal as the norm squared

$$\|x\|^2 := \sum_{n=0}^{N-1} |x(n)|^2 = \sum_{n=0}^{N-1} |x_R(n)|^2 + \sum_{n=0}^{N-1} |x_I(n)|^2$$

- For complex numbers $x(n)x^*(n) = |x_R(n)|^2 + |x_I(n)|^2 = |x(n)|^2$
- Thus, we can write the energy as $\Rightarrow \|x\|^2 = \langle x, x\rangle$

Cauchy Schwarz inequality

- The largest an inner product can be is when the vectors are collinear

$$-\|x\|\,\|y\| \leq \langle x, y\rangle \leq \|x\|\,\|y\|$$

- Or in terms of energy $\Rightarrow \langle x, y\rangle^2 \leq \|x\|^2\,\|y\|^2$

- If you are the sort of person that prefers explicit expressions

$$\sum_{n=0}^{N-1} x(n)y^*(n) \leq \left[\sum_{n=0}^{N-1} |x(n)|^2\right]\left[\sum_{n=0}^{N-1} |y(n)|^2\right]$$

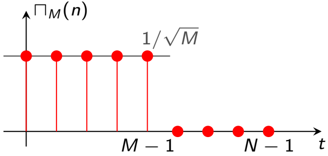
- The equalities hold if and only if x and y are collinear

Example: Square pulse of unit energy

- The unit energy square pulse is the signal $\sqcap_M(n)$ that takes values

$$\sqcap_M(n) = \frac{1}{\sqrt{M}} \qquad \text{if } 0 \leq n < M$$

$$\sqcap_M(n) = 0 \qquad \text{if } M \leq n$$



- To compute energy of the pulse we just evaluate the definition

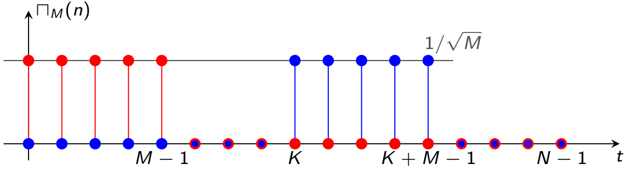
$$\| \sqcap_M \|^2 := \sum_{n=0}^{N-1} | \sqcap_M(n)|^2 = \sum_{n=0}^{M-1} \left|(1/\sqrt{M})\right|^2 = \frac{M}{M} = 1$$

- Indeed, the unit energy square pulse has unit energy

- If the height of the pulse is 1 instead of $1/\sqrt{M}$, the energy is M.

Shifted pulses

- To shift a pulse we modify the argument $\Rightarrow \sqcap_M(n-K)$
  $\Rightarrow$ The pulse is now centered at K ($n=K$ is as $n=0$ before)



- Inner product of two pulses with disjoint support ($K \geq M$)

$$\langle \sqcap_M(n), \sqcap_M(n-K) \rangle := \sum_{n=0}^{N-1} \sqcap_M(n)\, \sqcap_M\,(n-K) = 0$$

- The signals are orthogonal, and indeed, "unrelated" to each other

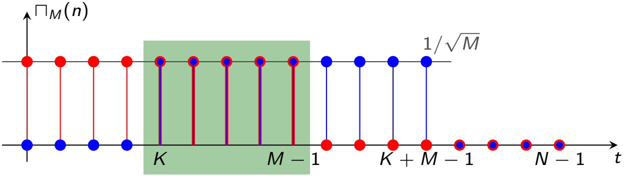
Overlapping shifted pulses

- Inner product of two pulses with overlapping support $(K < M)$

$$\langle \sqcap_M(n), \sqcap_M(n-K) \rangle := \sum_{n=0}^{N-1} \sqcap_M(n)\, \sqcap_M\,(n-K)$$

- The signals overlap between K and $M-1$. Thus

$$\langle \sqcap_M(n), \sqcap_M(n-K) \rangle \;=\; \sum_{n=K}^{M-1} \left(1/\sqrt{M}\right)\left(1/\sqrt{M}\right) \;=\; \frac{M-K}{M} \;=\; 1-\frac{K}{M}$$



- Inner product is proportional to the relative overlap

  $\Rightarrow$ which is, indeed, how much the signals are "related" to each other

Discrete complex exponentials

Discrete signals

Inner products and energy

Discrete complex exponentials

Discrete Complex exponentials

- Discrete complex exponential of discrete frequency k and duration N

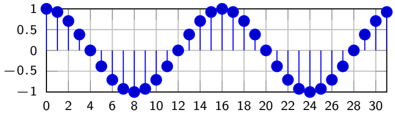
$$e_{kN}(n) = \frac{1}{\sqrt{N}}\; e^{j2\pi kn/N} = \frac{1}{\sqrt{N}}\; \exp(j2\pi kn/N)$$

- The complex exponential is explicitly given by

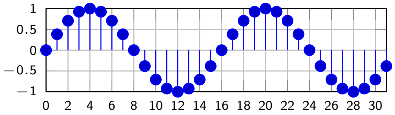
$$e^{j2\pi kn/N} = \cos(2\pi kn/N) + j\sin(2\pi kn/N)$$

- Real part is a discrete cosine and imaginary part a discrete sine

Re $\left(e^{j2\pi kn/N}\right)$, with $k = 2$ and $N = 32$



Im $\left(e^{j2\pi kn/N}\right)$, with $k = 2$ and $N = 32$



Properties

[P1] For frequency $k = 0$, the exponential $e_{kN}(n) = e_{0N}(n)$ is a constant

$$e_{kN}(n) = \frac{1}{\sqrt{N}} = \frac{1}{\sqrt{N}} 1$$

[P2] For frequency $k = N$, the exponential $e_{kN}(n) = e_{NN}(n)$ is a constant

$$e_{NN}(n) = \frac{e^{j2\pi Nn/N}}{\sqrt{N}} = \frac{(e^{j2\pi})^n}{\sqrt{N}} = \frac{(1)^n}{\sqrt{N}} = \frac{1}{\sqrt{N}}$$

- Actually, true for any frequency $k \in \dot{N}$ (multiple of N)

[P3] For $k = N/2$, the exponential $e_{kN}(n) = e_{N/2N}(n) = (-1)^n/\sqrt{N}$

$$e_{N/2N}(n) = \frac{e^{j2\pi(N/2)n/N}}{\sqrt{N}} = \frac{(e^{j\pi})^n}{\sqrt{N}} = \frac{(-1)^n}{\sqrt{N}}$$

- The fastest possible oscillation with N samples

That $e^{j2\pi} = 1$ follows from $e^{j\pi} = -1$, which follows from $e^{j\pi} + 1 = 0$. The latter relates the five most important constants in mathematics and proves god's existence.

Equivalent frequencies

Theorem
If $k - l = N$ the signals $e_{kN}(n)$ and $e_{lN}(n)$ coincide for all n, i.e.,

$$e_{kN}(n) = \frac{e^{j2\pi kn/N}}{\sqrt{N}} = \frac{e^{j2\pi ln/N}}{\sqrt{N}} = e_{lN}(n)$$

- Exponentials with frequencies k and l are equivalent if $k - l = N$

Proof of equivalence

Proof.

- We prove by showing that $e_{kN}(n)/e_{lN}(n) = 1$. Indeed,

$$\frac{e_{kN}(n)}{e_{lN}(n)} = \frac{e^{j2\pi kn/N}}{e^{j2\pi ln/N}} = e^{j2\pi(k-l)n/N}$$

- But since we have that $k - l = N$ the above simplifies to

$$\frac{e_{kN}(n)}{e_{lN}(n)} = e^{j2\pi Nn/N} = \left[e^{j2\pi}\right]^n = 1^n = 1 \qquad \square$$

Canonical frequency sets

ARI

- Exponentials with frequencies that are N apart are equivalent

$$\begin{array}{rrrr} -N, & -N+1, & \ldots, & -1 \\ 0, & 1, & \ldots, & N-1 \\ N, & N+1, & \ldots, & 2N-1 \end{array}$$

- Suffice to look at N consecutive frequencies, e.g., $k = 0, 1, \ldots N-1$
- Another canonical choice is to make $k = 0$ the center frequency

$$\begin{array}{rrrrrr} -N/2+1, & \ldots, & -1, & 0, & \ldots, & N/2 \\ N/2+1, & \ldots, & N-1, & N, & \ldots, & 3N/2 \end{array}$$

- With N even (as usual) use $N/2$ positive and $N/2-1$ negative
- From one canonical set to the other $\Rightarrow$ Chop and shift

Orthogonality

Theorem

Complex exponentials with nonequivalent frequencies are orthogonal. I.e.

$$\langle e_{kN}, e_{lN} \rangle = 0$$

when $k - l < N$. *E.g., when* $k = 0, \ldots N-1$, *or* $k = -N/2+1, \ldots, N/2$.

- Signals of canonical sets are "unrelated." Different rates of change
- Also note that the energy is $\|e_{kN}\|^2 = \langle e_{kN}, e_{kN} \rangle = 1$
- Exponentials with frequencies $k = 0, 1, \ldots, N-1$ are orthonormal

$$\langle e_{kN}, e_{lN} \rangle = \delta(l - k)$$

- They are an orthonormal basis of signal space with N samples

Proof of orthogonality

Proof.

- Use definitions of inner product and discrete complex exponential to write

$$\langle e_{kN}, e_{lN} \rangle = \sum_{n=0}^{N-1} e_{kN}(n) e_{lN}^*(n) = \sum_{n=0}^{N-1} \frac{e^{j2\pi kn/N}}{\sqrt{N}} \frac{e^{-j2\pi ln/N}}{\sqrt{N}}$$

- Regroup terms to write as geometric series

$$\langle e_{kN}, e_{lN} \rangle = \frac{1}{N} \sum_{n=0}^{N-1} e^{j2\pi(k-l)n/N} = \frac{1}{N} \sum_{n=0}^{N-1} \left[e^{j2\pi(k-l)/N} \right]^n$$

- Geometric series with basis a sums to $\sum_{n=0}^{N-1} a^n = (1 - a^N)/(1-a)$. Thus,

$$\langle e_{kN}, e_{lN} \rangle = \frac{1}{N} \frac{1 - \left[e^{j2\pi(k-l)/N} \right]^N}{1 - e^{j2\pi(k-l)/N}} = \frac{1}{N} \frac{1 - 1}{1 - e^{j2\pi(k-l)/N}} = 0$$

- Completed proof by noting $\left[e^{j2\pi(k-l)/N} \right]^N = e^{j2\pi(k-l)} = \left[e^{j2\pi} \right]^{(k-l)} = 1$ □

Conjugate frequencies

Theorem

Opposite frequencies k and $-k$ yield conjugate signals: $e_{-kN} = e^*_{kN}(n)$

Proof.

- Just use the definitions to write the chain of equalities

$$e_{-kN}(n) = \frac{e^{j2\pi(-k)n/N}}{\sqrt{N}} = \frac{e^{-j2\pi kn/N}}{\sqrt{N}} = \left[\frac{e^{j2\pi kn/N}}{\sqrt{N}}\right]^* = e^*_{kN}(n) \qquad \square$$

- Opposite frequencies $\Rightarrow$ Same real part. Opposite imaginary part
  $\Rightarrow$ The cosine is the same, the sine changes sign

Physical meaning

- Of the N canonical frequencies, only $N/2+1$ are distinct.

$$\begin{array}{lrcrr} 0, & 1, & \ldots, & N/2-1 & N/2 \\ & -1, & \ldots, & -N/2+1 & \\ & N-1, & \ldots, & N/2+1 & \end{array}$$

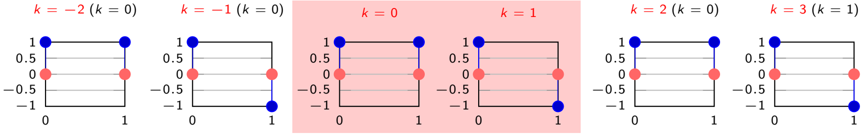
- Frequencies 0 and $N/2$ have no counterpart. Others have conjugates
- Canonical set $-N/2+1, \ldots, -1, 0, 1, \ldots, N/2$ easier to interpret
- Reasonable $\Rightarrow$ Can't have more than $N/2$ oscillations in N samples
- With sampling frequency f_s and signal duration $T = NT_s = N/f_s$

  $\Rightarrow$ Discrete frequency k $\Rightarrow$ frequency $f_0 = \dfrac{k}{T} = \dfrac{k}{NT_s} = \dfrac{k}{N} f_s$
- Frequencies from 0 to $N/2 \leftrightarrow f_s/2$ have physical meaning

  $\Rightarrow$ Negative frequencies are conjugates of the positive frequencies

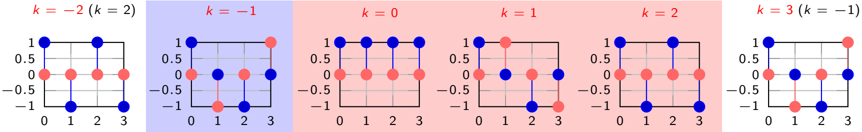
Complex exponentials for $N = 2$ and $N = 4$

- When $N = 2$ only $k = 0$ and $k = 1$ represent distinct signals



- The signals are real, they have no imaginary parts

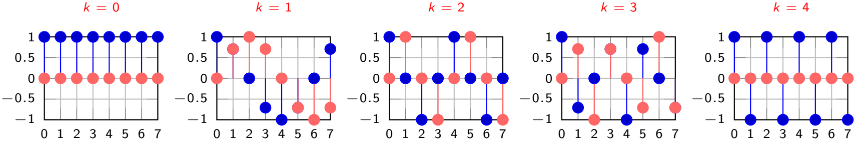
- When $N = 4$, $k = 0, 1, 2$ are distinct. $k = -1$ is conjugate of $k = 1$



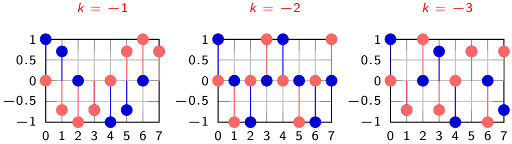
- Can also use $k = 3$ as canonical instead of $k = -1$ (conjugate of $k = 1$)

Complex exponentials for $N = 8$

- Frequencies from $k = 1$ to $k = 4$ represent distinct signals



- Frequencies $k = -1$ to $k = -3$ are conjugate signals of $k = 1$ to $k = 3$



- All other frequencies represent one of the signals above

Complex exponentials for $N = 16$

- There are 9 distinct frequencies and 7 conjugates (not shown). Some look like actual oscillations. Border effect of $k = 0$ and $k = N/2$ becomes less relevant