Fourier transforms

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Discrete signals and DFT

- Fourier analysis of discrete signals $x : [0, N-1] \to \mathbb{C}$ $\Rightarrow$ DFT, iDFT
- Good (and quick) computational tool
  - $\Rightarrow$ Signal analysis $\Rightarrow$ pattern discovery, frequency components
  - $\Rightarrow$ Signal processing $\Rightarrow$ compression, noise removal
- Two important limitations
  - $\Rightarrow$ Time is neither discrete nor finite (not always, at least)
  - $\Rightarrow$ Properties and interpretations are easier in continuous time
- Fourier analysis of continuous signals $\Rightarrow$ Fourier transform (FT)

Continuous time signals

To infinity and beyond

- We have been dealing with discrete signals $x : [0, N-1] \rightarrow \mathbb{C}$
- To infinity $\Rightarrow$ Let number of samples go to infinity
  $\Rightarrow$ Discrete time signal $x : \mathbb{Z} \rightarrow \mathbb{C}$
  $\Rightarrow$ Values $x(n)$ for $n = \ldots, -1, 0, 1, \ldots$
- And beyond $\Rightarrow$ Fill in the gaps between samples
  $\Rightarrow$ Continuous time signal $x : \mathbb{R} \rightarrow \mathbb{C}$
  $\Rightarrow$ Values $x(t)$ for t any real number in $(-\infty, +\infty)$
- Let's begin by studying continuous time signals

Continuous time signals

- Continuous time variable $t \in \mathbb{R}$.
- Continuous time signal x is a function that maps t to real value $x(t)$

$$x : \mathbb{R} \to \mathbb{R}$$

- The values that the signal takes at time t is $x(t)$
- It will make sense to talk about complex signals (as in discrete case)

$$x : \mathbb{R} \to \mathbb{C}$$

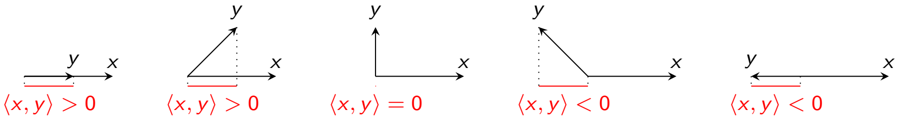
- where the values $x(t) = x_R(t) + j\, x_I(t)$ are complex numbers

Inner product

- Given two signals x and y define the inner product of x and y as

$$\langle x, y \rangle := \int_{-\infty}^{\infty} x(t) y^*(t) dt$$

- Akin to inner product of discrete signals $\Rightarrow \langle x, y \rangle = \sum_{n=0}^{N} x(n) y(n)$



- But we have infinite number of components. To infinity and beyond
- Intuition holds $\Rightarrow \langle x, y \rangle$ is how much of y falls in x direction
- E.g., if $\langle x, y \rangle = 0$ the signals are orthogonal. They are "unrelated"

Norm and energy

- As for regular (finite dimensional) signals define the norm of signal x

$$\|x\| := \left[\int_{-\infty}^{\infty} |x(t)|^2 dt\right]^{1/2} = \left[\int_{-\infty}^{\infty} |x_R(t)|^2 dt + \int_{-\infty}^{\infty} |x_I(t)|^2 dt\right]^{1/2}$$

- More important, define the energy of the signal as the norm squared

$$\|x\|^2 := \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |x_R(t)|^2 dt + \int_{-\infty}^{\infty} |x_I(t)|^2 dt$$

- For complex numbers $x(t)x^*(t) = |x_R(t)|^2 + |x_I(t)|^2 = |x(t)|^2$
- Thus, we can write the energy as $\Rightarrow \|x\|^2 = \langle x, x\rangle$
- Energy might be infinite. When energy is finite we write $\|x\|^2 < \infty$

Cauchy Schwarz inequality

- The largest an inner product can be is when the vectors are collinear

$$-\|x\|\,\|y\| \leq \langle x, y\rangle \leq \|x\|\,\|y\|$$

- Or in terms of energy $\Rightarrow \langle x, y\rangle^2 \leq \|x\|^2\,\|y\|^2$
- If you are the sort of person that prefers explicit expressions

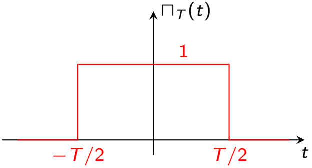
$$\int_{-\infty}^{\infty} x(t)y^*(t)dt \leq \left[\int_{-\infty}^{\infty} |x(t)|^2 dt\right]\left[\int_{-\infty}^{\infty} |y(t)|^2 dt\right]$$

- The equalities hold if and only if x and y are collinear

Example: Square pulse

- The square pulse is the signal $\sqcap_T(t)$ that takes values

$$\sqcap_T(t) = 1 \qquad \text{for } -\frac{T}{2} \leq t < \frac{T}{2}$$
$$\sqcap_T(t) = 0 \qquad \text{otherwise}$$



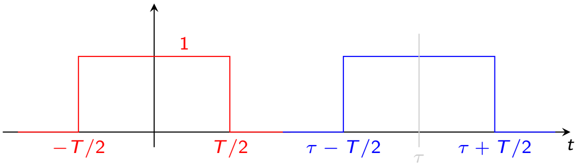
- To compute energy of the pulse we just evaluate the definition

$$\|\sqcap_T(t)\|^2 := \int_{-\infty}^{\infty} |\sqcap_T(t)(t)|^2 dt = \int_{-T/2}^{T/2} |1|^2 \, dt = T$$

- Energy proportional to pulse duration (duh!)
- Can normalize energy dividing by $\sqrt{T}$. But we rather not.

- To shift a pulse we modify the argument $\Rightarrow \sqcap_T(t-\tau)$

  $\Rightarrow$ The pulse is now centered at τ ($t=\tau$ is as $t=0$ before)



- Inner product of two pulses with disjoint support ($\tau > T$)

$$\langle \sqcap_T(t), \sqcap_T(t-\tau) \rangle := \int_{-\infty}^{\infty} \sqcap_T(t)\, \sqcap_T\,(t-\tau) = 0$$

- The signals are orthogonal, and indeed, "unrelated" to each other

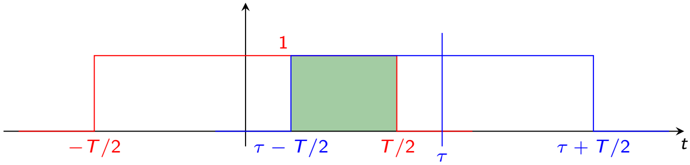
Shifted pulses (2 of 2)

- Inner product of two pulses with overlapping support ($\tau > T$)

$$\langle \sqcap_T(t), \sqcap_T(t-\tau) \rangle := \int_{-\infty}^{\infty} \sqcap_T(t) \sqcap_T(t-\tau)$$

- The signals overlap between $\tau - T/2$ and $T/2$. Thus

$$\langle \sqcap_T(t), \sqcap_T(t-\tau) \rangle = \int_{\tau - T/2}^{T/2} (1)(1) dt = \frac{T}{2} - \left(\tau - \frac{T}{2}\right) = T - \tau$$



- Inner product is proportional to the relative overlap

  $\Rightarrow$ which is, indeed, how much the signals are “related” to each other

Complex exponentials

- Inner product and energy are indefinite integrals $\Rightarrow$ need not exist
- Complex exponential of frequency f is e_f with $e_f(t) = e^{j2\pi ft}$
- Since they have unit modulus ($|e_f(t)| = |e^{j2\pi ft}| = 1$), their energy is

$$\|e_f\|^2 := \int_{-\infty}^{\infty} |e_f(t)|^2 dt = \int_{-\infty}^{\infty} 1 dt = \infty$$

- Inner product of complex exponentials not defined ("keeps oscillating")

$$\langle e_f, e_g \rangle := \int_{-\infty}^{\infty} e_f(t) e_g^*(t) dt = \int_{-\infty}^{\infty} e^{j2\pi ft} e^{-j2\pi gt} dt = \int_{-\infty}^{\infty} e^{j2\pi(f-g)t} dt \Rightarrow \nexists$$

- This is a problem because we can't talk about orthogonality
  $\Rightarrow$ Still, a complex exponential is much more like itself than another

Fourier transform

Continuous time signals

Fourier transform

Inverse Fourier transform

Delta function

Generalized orthogonality

Generalized Fourier transforms

Properties of the Fourier transform

Convolution

Definition of Fourier transform

- The Fourier transform of x is the function $X : \mathbb{R} \to \mathbb{C}$ with values

$$X(f) := \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt$$

- We write $X = \mathcal{F}(x)$. All values of X depend on all values of x
- Integral need not exist $\Rightarrow$ Not all signals have a Fourier transform
- The argument f of the Fourier transform is referred to as frequency
- Or, define e_f with values $e_f(t) = e^{j2\pi f t}$ to write as inner product

$$X(f) = \langle x, e_f \rangle = \int_{-\infty}^{\infty} x(t) e_f^*(t) dt$$

- Both, time and frequency are real $\Rightarrow$ domain is infinite and dense
  $\Rightarrow$ This is an analytical tool, not a computational tool (as the DFT)

Example: Fourier transform of a square pulse

- Since pulse is not null only when $T/2 \leq t \leq T/2$ we reduce $X(f)$ to

$$X(f) := \int_{-\infty}^{\infty} \sqcap_T(t) e^{-j2\pi f t} dt = \int_{-T/2}^{T/2} e^{-j2\pi f t} dt$$

- For $f \neq 0$, the primitive of $e^{-j2\pi f t}$ is $(-1/j2\pi f)e^{-j2\pi f t}$, which yields

$$X(f) = \left[\frac{-e^{-j2\pi f T/2}}{j2\pi f} - \frac{-e^{+j2\pi f T/2}}{j2\pi f}\right] = \frac{\sin(\pi f T)}{\pi f}$$

- Where we used $e^{j\pi f T} - e^{-j\pi f T} = 2j \sin(\pi f T)$

- For $f = 0$ we have $e^{-j2\pi f t} = 1$ and $X(f)$ reduces to $\Rightarrow X(f) = T$

The sinc function

- Transform is important enough to justify definition of sinc function

$$\text{sinc}(u) = \frac{\sin(u)}{u} \qquad \text{for } u \neq 0$$
$$\text{sinc}(u) = 1 \qquad \text{for } u = 0$$

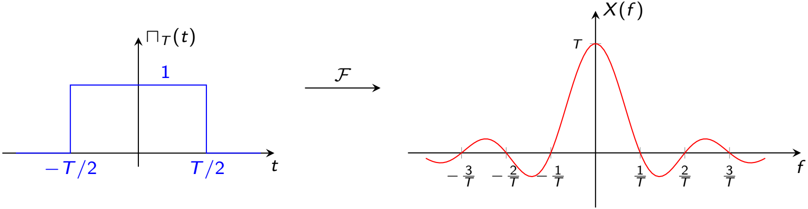
- Value at origin, $\text{sinc}(0) = 1$, makes the function continuous
- With this definition and $f \neq 0$ we can write the pulse transform as

$$X(f) = \frac{\sin(\pi f T)}{\pi f} = T\frac{\sin(\pi f T)}{\pi f T} = T\text{sinc}(\pi f T)$$

- Which is also true for $f = 0$ because $X(0) = T\text{sinc}(\pi 0 T) = T$

The pulse and its transform

- Fourier transform of pulse of width T is sinc with null crossings $\dfrac{k}{T}$

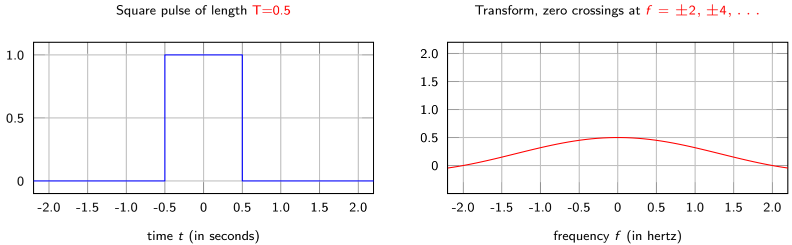


- Most of the Fourier Transform energy is between $-1/T$ and $1/T$

$$\int_{-1/T}^{1/T} \left|X(f)\right|^2 df = \int_{-1/T}^{1/T} \left|T\text{sinc}(\pi f T)\right|^2 df \approx 0.90T = 0.90\| \sqcap_T(t)\|^2$$

Pulses of different width

- Transforms of wider pulses are more concentrated around $f = 0$

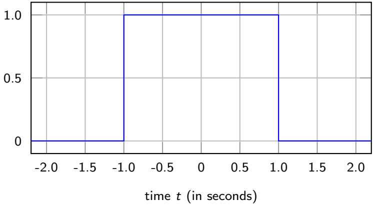


- Consistent with interpretation that shorter pulses are faster varying

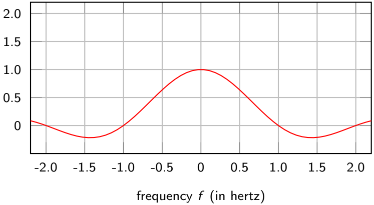
Pulses of different width

- Transforms of wider pulses are more concentrated around $f = 0$

Square pulse of length T=1



Transform, zero crossings at $f = \pm 1, \pm 2, \ldots$



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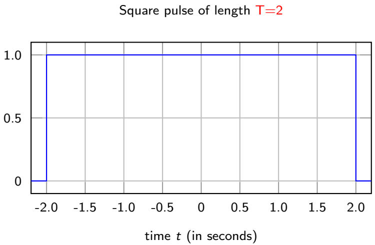
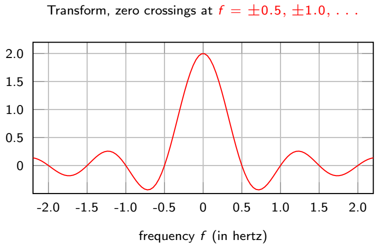




- Consistent with interpretation that shorter pulses are faster varying

The Fourier transform and the DFT

- Let's compute a Fourier transform by approximating the integral
- Use samples spaced by T_s time units

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi f t}dt \approx T_s \sum_{-\infty}^{\infty} x(nT_s)e^{-j2\pi f nT_s}$$

- Still not computable $\Rightarrow$ consider only N samples from 0 to $N-1$

$$X(f) \approx T_s \sum_{k=0}^{N-1} x(nT_s)e^{-j2\pi f nT_s}$$

- This is true for all frequencies. Consider frequencies $f = (k/N)f_s$

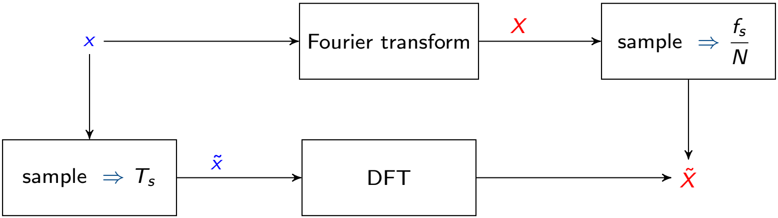
$$X\left(\frac{k}{N}f_s\right) \approx T_s \sum_{k=0}^{N-1} x(nT_s)e^{-j2\pi (k/N)f_s nT_s} = T_s \sum_{k=0}^{N-1} x(nT_s)e^{-j2\pi kn/N}$$

- Definition of the DFT of a discrete signal (up to constants)

DFT as approximation of Fourier transform

- Define $\tilde{x}$ with $\tilde{x}(n) = x(nT_s)$. The DFT of $\tilde{X} = \mathcal{F}(\tilde{x})$ has components

$$\tilde{X}(k) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \tilde{x}(n) e^{-j2\pi kn/N} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} x(nT_s) e^{-j2\pi kn/N} = \frac{1}{T_s\sqrt{N}} X\left(\frac{k}{N} f_s\right)$$



- Can then aproximate Fourier transform as $\Rightarrow X\left(\frac{k}{N} f_s\right) \approx T_s\sqrt{N}\tilde{X}(k)$
- Approximation becomes equality at infinity and beyond $(N \to \infty,\ T_s \to 0)$

Fourier transform of a complex exponential

- Complex exponential of frequency f_0 $\Rightarrow$ $e_{f_0}(t) = e^{j2\pi f_0 t}$
- Use inner product form to write the components of $X = \mathcal{F}(e_{f_0})$ as

$$X(f) = \langle x, e_f \rangle = \langle e_{f_0}, e_f \rangle$$

- We've seen that $\langle e_{f_0}, e_f \rangle = \infty$ if $f = f_0$ and oscillates ($\nexists$) if $f \neq f_0$
- The complex exponential does not have a Fourier transform

  $\Rightarrow$ Happens because energy of complex exponentials is not finite
- Truncate to $T/2 \leq t \leq T/2$ $\Rightarrow$ multiply by square pulse $\sqcap_T(t)$

$$\tilde{e}_{f_0 T}(t) := e_{f_0}(t) \sqcap_T (t) = e^{j2\pi f_0 t} \sqcap_T (t)$$

Fourier transform of a complex exponential

- Truncated exponential not null only when $T/2 \leq t \leq T/2$ (pulse)
- Then, the Fourier transform $\tilde{X}_T(f) := \mathcal{F}(\tilde{e}_{f_0 T})$ is given by

$$\tilde{X}(f) := \int_{-\infty}^{\infty} e^{j2\pi f_0 t} \sqcap_T(t) e^{-j2\pi f t} dt = \int_{-T/2}^{T/2} e^{j2\pi f_0 t} e^{-j2\pi f t} dt = \int_{-T/2}^{T/2} e^{-j2\pi (f-f_0) t} dt$$

- Same as pulse transform, except for frequency shift in exponent
- For $f \neq f_0$, primitive of $e^{-j2\pi f t}$ is $(-1/j2\pi(f-f_0))e^{-j2\pi(f-f_0)t}$. Thus

$$\tilde{X}(f) = \left[\frac{-e^{-j2\pi(f-f_0)T/2}}{j2\pi(f-f_0)} - \frac{-e^{+j2\pi(f-f_0)T/2}}{j2\pi(f-f_0)}\right] = \frac{\sin(\pi(f-f_0)T)}{\pi(f-f_0)}$$

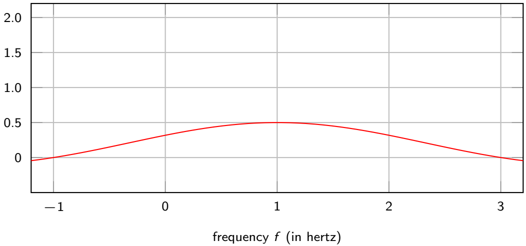
- For $f = f_0$ we have $e^{-j2\pi(f-f_0)t} = 1$ and $\tilde{X}(f)$ reduces to $\Rightarrow \tilde{X}(f) = T$

Shifted sinc

- Fourier transform of truncated complex exponential is shifted sinc

$$\tilde{X}(f) = T\mathrm{sinc}(\pi(f - f_0)T)$$

Transform, (centered at frequency $f_0 = 1$)



- As $T \to \infty$ truncated exponential approaches exponential
  - $\Rightarrow$ And shifted sinc becomes infinitely tall $\Rightarrow$ delta function

Shifted sinc

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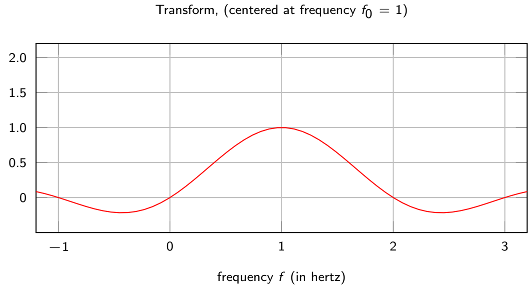
$$\tilde{X}(f) = T\text{sinc}(\pi(f - f_0)T)$$



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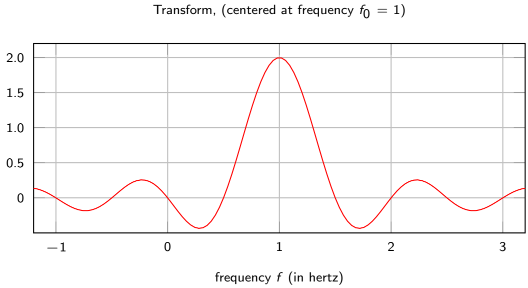

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Inverse Fourier transform

Continuous time signals

Fourier transform

Inverse Fourier transform

Delta function

Generalized orthogonality

Generalized Fourier transforms

Properties of the Fourier transform

Convolution

Inverse Fourier transform

- Given a transform X, the inverse Fourier transform is defined as

$$x(t) := \int_{-\infty}^{\infty} X(f) e^{j2\pi f t}\, df$$

- We denote the inverse transform as $x = \mathcal{F}^{-1}(X)$
- Sign in the exponent changes with respect to Fourier transform

- Can write as inner product $\Rightarrow x(t) = \langle X, e_{-t} \rangle$ $(e_{-t}(f) = e^{-j2\pi ft})$
- As in the case of the iDFT, this is not the most useful interpretation

Indeed, the inverse of the Fourier transform

Theorem

The inverse Fourier transform $\tilde{x}$ of the Fourier transform X of a given signal x is the given signal x

$$\tilde{x} = \mathcal{F}^{-1}(X) = \mathcal{F}^{-1}[\mathcal{F}(x)] = x$$

- Signals with Fourier transforms can be written as sums of oscillations

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft}\, df \approx (\Delta f) \sum_{n=\infty}^{\infty} X(f_n) e^{j2\pi f_n t}$$

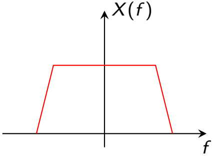
- This is conceptual, not literal (as was the case in discrete signals)

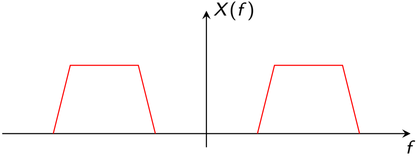
Frequency decomposition of a signal

- $X(f)$ determines the density of frequency f in the signal $x(t)$

$$x(t) \approx \sum_{n=\infty}^{\infty} (\Delta f) X(f_n) e^{j2\pi f_n t}$$

- It represents relative contribution (as opposed to absolute)





- Signal on left accumulates mass at low frequencies (changes slowly)
- Signal on right accumulates mass at high frequencies (changes fast)

Proof of inverse Fourier transform

Proof.

- We want to show $\Rightarrow \tilde{x} = \mathcal{F}^{-1}(X) = \mathcal{F}^{-1}[\mathcal{F}(x)] = x$. Use definitions
- From definition of inverse transform of $X \Rightarrow \tilde{x}(\tilde{t}) := \int_{-\infty}^{\infty} X(f) e^{j2\pi f \tilde{t}}\, df$
- From definition of transform of $x \Rightarrow X(f) := \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt$
- Substituting expression for $X(f)$ into expression for $\tilde{x}(\tilde{t})$ yields

$$\tilde{x}(\tilde{t}) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt \right] e^{j2\pi f \tilde{t}}\, df$$

- Repeating steps done for DFT and iDFT with integrals instead of sums

Proof of inverse Fourier transform

Proof.

- Exchange integration order to integrate first over f and then over t

$$\tilde{x}(\tilde{t}) = \int_{-\infty}^{\infty} x(t) \left[\int_{-\infty}^{\infty} e^{j2\pi f \tilde{t}} e^{-j2\pi f t} df \right] dt$$

- Pulled $x(t)$ out because it doesn't depend on k

- Innermost integral is the inner product between $e_{\tilde{t}}$ and e_t.

$$\int_{-\infty}^{\infty} e^{j2\pi f \tilde{t}} e^{-j2\pi f t} df = \langle e_{\tilde{t}}, e_t \rangle$$

- Up until now we repeated same steps we did for DFT and iDFT
- But we encounter a problem $\Rightarrow \langle e_{\tilde{t}}, e_t \rangle$ does not exist (infinity, oscillates)

- To exchange integration order, all integrals have to exist. But one doesn't
  $\Rightarrow$ It is mathematically incorrect to interchange the order of integration

Proof of inverse Fourier transform

Proof.

- Replace infinite summation boundaries with finite summation boundaries

$$\tilde{x}(\tilde{t}) \overset{F\to\infty}{=} \int_{-\infty}^{\infty} x(t) \left[\int_{-F/2}^{F/2} e^{j2\pi f \tilde{t}} e^{-j2\pi f t} df \right] dt$$

- Eventually, we need to take $F \to \infty$, but not yet.
- All integrals exist now. Innermost one is a sinc (truncated exponential)

$$\int_{-F/2}^{F/2} e^{j2\pi f \tilde{t}} e^{-j2\pi f t} df = F\text{sinc}(\pi(t - \tilde{t})F)$$

- Substitute sinc for innermost integral on previous expression

$$\tilde{x}(\tilde{t}) \overset{F\to\infty}{=} \int_{-\infty}^{\infty} x(t) \Big[F\text{sinc}(\pi(t - \tilde{t})F) \Big] dt$$

Proof of inverse Fourier transform

Proof.

- take the limit formally $\Rightarrow \tilde{x}(\tilde{t}) = \lim_{F\to\infty} \int_{-\infty}^{\infty} x(t) \Big[F\mathrm{sinc}(\pi(t-\tilde{t})F) \Big] dt$
- The sinc function is centered at time $t = \tilde{t}$
- The sinc becomes infinitely tall and thin as we take $F \to \infty$
- Can then take $x(\tilde{t})$ outside of the integral (only "meaningful" value)

$$\tilde{x}(\tilde{t}) = \lim_{F\to\infty} x(\tilde{t}) \int_{-\infty}^{\infty} F\mathrm{sinc}(\pi(t-\tilde{t})F) dt$$

- The sinc function has unit integral $\Rightarrow \int_{-\infty}^{\infty} F\mathrm{sinc}(\pi(t-\tilde{t})F) = 1$
- We then have $\tilde{x}(\tilde{t}) = x(\tilde{t})$ and $\tilde{x} = x$ as we wanted to show □

Fourier transform pairs

- Symmetry between transform and inverse $\Rightarrow$ Transform pairs
- Interpret given function z as signal. Fourier transform $X = \mathcal{F}(z)$ is

$$X(f) = \int_{-\infty}^{\infty} z(t) e^{-j2\pi f t} dt$$

- Conjugate z and interpet z^* as a transform. Inverse $x = \mathcal{F}^{-1}(z^*)$ is

$$x(t) = \int_{-\infty}^{\infty} z^*(f) e^{j2\pi f t}\, df = \left[\int_{-\infty}^{\infty} z(f) e^{-j2\pi f t}\, df \right]^*$$

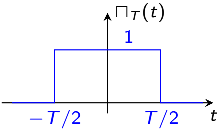
- Same integrals except for switch of integration index and argument

$$X(f) = x^*(t), \qquad \text{when } f = t$$

- X is transform of z and z is transform of $X^* \equiv x^*$ $\Rightarrow$ They are a pair

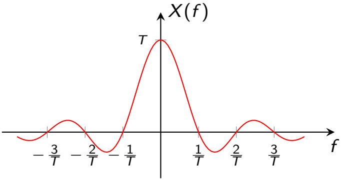
$\Rightarrow$ Conjugation unnecessary when signal and transform are real

- Square of length T $\Rightarrow$ Sinc with zero crossings at k/T, $T sinc(\pi f T)$

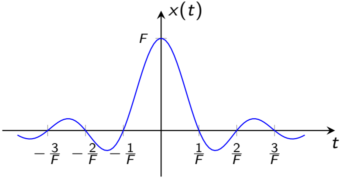


$\xrightarrow{\mathcal{F}}$

$\xleftarrow{\mathcal{F}^{-1}}$

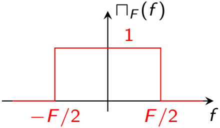


- Sinc with zero crossings at k/F, $T sinc(\pi F t)$ $\Rightarrow$ Square of length F



$\xrightarrow{\mathcal{F}}$

$\xleftarrow{\mathcal{F}^{-1}}$



- Transform of sinc pulse is difficult to compute through direct operation

Delta function

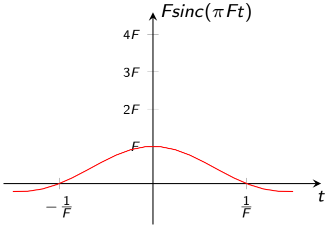
Sequence of progressively taller sinc pulses

- Define the continuous time delta function as the limit of a sinc pulse

$$\delta(t) := \lim_{F \to \infty} F sinc(\pi F t)$$

- Limit is $\delta(t) = \infty$ for $t = 0$
- But does not exist for other t

  $\Rightarrow$ Oscillates between $\pm 1/\pi t$



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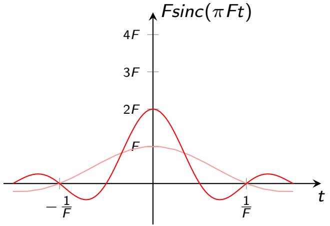

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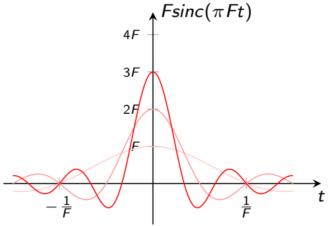

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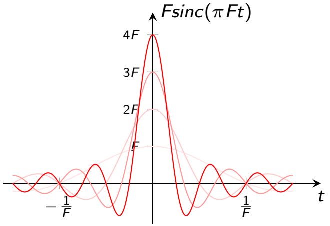

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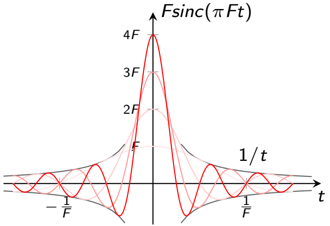

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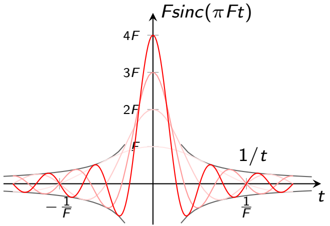


- Limit is $\delta(t) = \infty$ for $t = 0$
- But does not exist for other t
  $\Rightarrow$ Oscillates between $\pm 1/\pi t$

- On second thought, maybe we should use a different definition

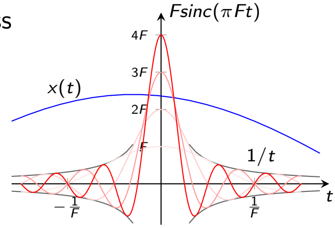
- Intuitively, we want to say that the delta function is
  $\Rightarrow$ Infinity for $t = 0$ $\Rightarrow$ $\delta(t) = \infty$ for $t = 0$
  $\Rightarrow$ Null for all other t $\Rightarrow$ $\delta(t) = 0$ for $t \neq 0$
- But the question is what can we say mathematically? $\Rightarrow$ Integrate

Limit of inner products

- Integrate the product of a signal with a sinc that is thin and tall
  $\Rightarrow$ Recovers the value of the signal at time $t = 0$

- Since $x(0)$ multiplies most of sinc mass

$$\int_{-\infty}^{\infty} x(t) F sinc(\pi F t) dt \approx x(0)$$



- Can write formally as

$$\lim_{F \to \infty} \int_{-\infty}^{\infty} x(t) F sinc(\pi F t) dt = x(0)$$

- Observe that integral is the inner product of x with sinc. Then

$$\lim_{F \to \infty} \langle x, F sinc(\pi F t) \rangle = x(0)$$

- Inner product of a signal with arbitrarily tall sinc is its value at zero

Delta function

- Define delta function as the entity δ that has this property. I.e., if

$$\langle x, \delta \rangle = x(0)$$

- for any signal x, we say that δ is a delta function

- In terms of integrals we write $\Rightarrow \displaystyle\int_{-\infty}^{\infty} x(t)\delta(t)dt = x(0)$

- Is the delta function a function? $\Rightarrow$ Of course not
- We say that δ is a distribution or generalized function

- Abstract entity without meaning until we pass through an integral
  $\Rightarrow$ Can't observe directly, but can observe its effect on other signals

- Can define orthogonality and transforms of complex exponentials

Generalized orthogonality

Orthogonality of complex exponentials

- Consider complex exponentials of frequencies f and g
  $\Rightarrow$ Frequency $f \;\Rightarrow e_f(t) = e^{j2\pi ft}$. Frequency $g \;\Rightarrow e_g(t) = e^{j2\pi gt}$
- We define their inner product $\langle e_f, e_g \rangle$ as the delta function $\delta(f-g)$

$$\langle e_f, e_g \rangle = \delta(f-g)$$

- This is a definition, not a derivation. We are accepting it to be true.
- If it is a definition: Does it make sense? What's its meaning?

It makes sense

- Complex exponentials don't have a mutual inner product.
- But truncated exponentials $e_{f,T}$ and e_{gT} do have a mutual product

  $\Rightarrow$ Multiply by $\sqcap_T$. Make signal null for $t > T/2$ and $t < T/2$

- Can write inner product of truncated signals as

$$\langle e_{fT}, e_{gT} \rangle := \int_{-T/2}^{T/2} e_f(t) e_g^*(t) dt = \int_{-T/2}^{T/2} e^{j2\pi ft} e^{-j2\pi gt} dt = \int_{-T/2}^{T/2} e^{j2\pi(f-g)t} dt$$

- Integral above resolves to a sinc with zero crossings at k/T

$$\langle e_{fT}, e_{gT} \rangle = T\text{sinc}\big[\pi(f-g)T\big]$$

- As $T \to \infty$ truncated signals approach non-truncated counterparts...
- ...and the sinc limit is our first attempt at defining $\delta(f-g)$
- Definition didn't work. But we are looking for sense, not meaning

What does it mean?

- Delta function is not observable directly, only after integration
- For an arbitrary given signal $X(f)$ we must have

$$\int_{-\infty}^{\infty} X(f)\langle e_{fT}, e_{gT}\rangle df = \int_{-\infty}^{\infty} X(f)\delta(f-g)df = X(g)$$

- Equivalently, we can write in terms of integrals

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(f)e^{j2\pi ft}e^{-j2\pi gt}\,dt\,df = X(g)$$

- OK, fine, but really, stop messing and tell us what it means

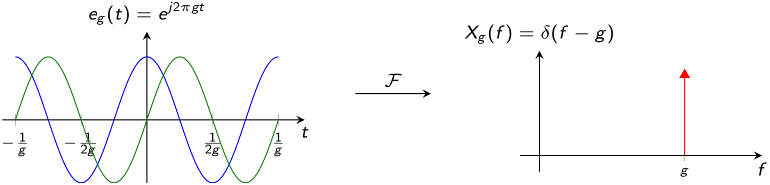
  $\Rightarrow$ When $f = g$ $\Rightarrow \langle e_f, e_f\rangle = \infty$. When $f \neq g$ $\Rightarrow \langle e_f, e_g\rangle = 0$
- Can use for intuitive reasoning, but not for mathematical derivations

Generalized Fourier transforms

Fourier transform of complex exponential

- Again, we can define, not derive, the Fourier transform of e_g
- Denote as $X_g := \mathcal{F}(e_g)$ the transform of e_g. We define X_g as

$$X_g(f) = \delta(f - g)$$



- We draw delta functions with an arrow pointing to the sky

It makes sense and it has meaning

- Does it make sense to have $X_g(f) = \delta(f-g)$
- Yes $\Rightarrow$ Transform definition consistent with orthogonality definition

$$X_g(f) = \langle e_g, e_f \rangle = \delta(f-g)$$

- Yes $\Rightarrow$ Definition is consistent with definition of inverse transform

$$e_g(t) = \int_{-\infty}^{\infty} X_g(f) e^{j2\pi ft} df = \int_{-\infty}^{\infty} \delta(f-g) e^{j2\pi ft} df = e^{j2\pi gt}$$

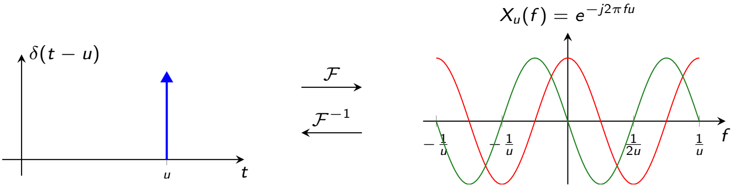
- Making $X_g(f) = \delta(f-g)$ maintains Fourier analysis coherence

- Definition has clear, albeit, disappointingly trivial meaning
- Exponential of freq. g can be written as exponential of freq. g

Fourier transform of a shifted delta function

- Denote as X_u the transform of the shifted delta function $\delta(t-u)$
- This one we can compute $\Rightarrow$ Complex exponential of frequency u

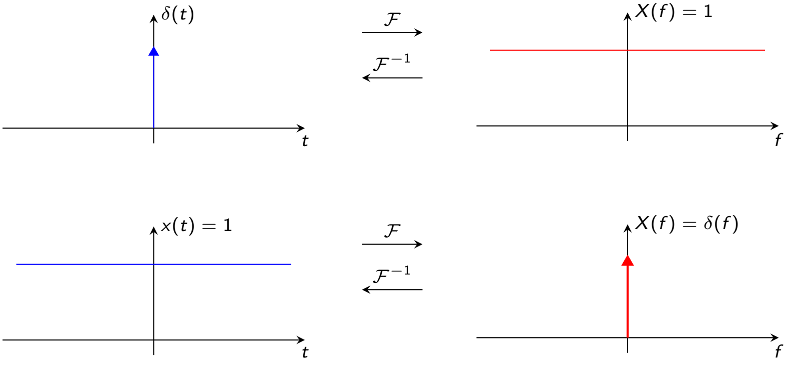
$$X_u(f) = \int_{-\infty}^{\infty} \delta(t-u) e^{-j2\pi ft} dt = e^{-j2\pi fu} = e_{-u}(f)$$



- It is the inverse we need to define as a delta function centered at u

The delta – constant transform pair

- When frequencies are null we have constants and unshifted deltas
- Transform of $x(t) = \delta(t) \;\Rightarrow X(f) = 1$. Transform of $x(t) = 1 \;\Rightarrow X(f) = \delta(f)$



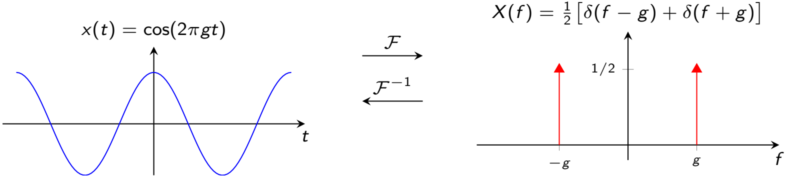
Fourier transform of a cosine

- To find Fourier transform of cosine write as difference of exponentials

$$\cos(2\pi g t) = \frac{1}{2}\left[e^{j2\pi g t} + e^{-j2\pi g t}\right]$$

- Since Fourier is a linear operator we transform each of the summands

$$X(f) = \frac{1}{2}\left[\delta(f-g) + \delta(f+g)\right]$$



- Pair of deltas of "height 1/2" at (opposite) frequencies $\pm g$

Properties of the Fourier transform

Three properties we already studied for the DFT

- Fourier transform is conjugate symmetric, linear, and conserves energy
- Transforms of real signals satisfy $\Rightarrow X(-k) = X^*(k)$
- Linearity $\Rightarrow \mathcal{F}(ax + by) = a\mathcal{F}(x) + b\mathcal{F}(y)$
- Energy $\Rightarrow \displaystyle\int_{-\infty}^{\infty} \big|x(t)\big|^2 dt = \big\|x\big\|^2 = \big\|X\big\|^2 = \int_{-\infty}^{\infty} \big|X(f)\big|^2 df$
- Not surprising, Fourier transform and DFT are conceptually identical
- Properties follow from properties of inner products and orthogonality
- Both transforms are projections on complex exponentials (inner product)
- And both project onto sets of orthogonal signals

Symmetry

Theorem

The Fourier transform $X = \mathcal{F}(x)$ of a real signal x is conjugate symmetric

$$X(-f) = X^*(f)$$

- For real signals only positive half of spectrum carries information
- Conjugate symmetry implies that $X(-f)$ and $X^*(f)$ are such that...
  - $\Rightarrow$ Real parts are equal $\Rightarrow \mathrm{Re}\,(X(f)) = \mathrm{Re}\,(X(-f))$
  - $\Rightarrow$ Imaginary parts are opposites $\Rightarrow \mathrm{Im}\,(X(f)) = \mathrm{Im}\,(X(-f))$
  - $\Rightarrow$ Moduli are equal $\Rightarrow |X(f)| = |X(-f)|$

Proof of symmetry property

Proof.

- Write the Fourier transform $X(-k)$ using its definition

$$X(-f) := \int_{-\infty}^{\infty} x(t) e^{-j2\pi(-f)t} dt$$

- When the signal is real, its conjugate is itself $\Rightarrow x(n) = x^*(n)$
- Conjugating a complex exponential $\Rightarrow$ changing the exponent's sign
- Can then rewrite $\Rightarrow X(-f) := \int_{-\infty}^{\infty} x^*(t) \left(e^{-j2\pi ft}\right)^* dt$
- Integration and multiplication can change order with conjugation

$$X(-f) = \left[\int_{-\infty}^{\infty} x^*(t) \left(e^{-j2\pi ft}\right)^* dt\right]^* = X^*(f) \qquad \square$$

Linearity

Theorem

The Fourier transform of a linear combination of signals is the linear combination of the respective Fourier transforms of the individual signals,

$$\mathcal{F}(ax + by) = a\mathcal{F}(x) + b\mathcal{F}(y).$$

Proof.

- Let $Z := \mathcal{F}(ax + by)$. From the Fourier transform definition

$$Z(f) = \int_{-\infty}^{\infty} \Big[ax(t) + by(t)\Big] e^{-j2\pi ft} dt$$

- Expand the product, reorder terms, identify transforms of x and y

$$Z(f) = a\int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt + b\int_{-\infty}^{\infty} y(t)e^{-j2\pi ft}dt = aX(f) + bY(f) \quad \square$$

Energy conservation

Theorem (Parseval)

Let $X = \mathcal{F}(x)$ be the Fourier transform of signal x. The energies of x and X are the same, i.e.,

$$\int_{-\infty}^{\infty} \left|x(t)\right|^2 dt = \left\|x\right\|^2 = \left\|X\right\|^2 = \int_{-\infty}^{\infty} \left|X(f)\right|^2 df$$

- It follows that $X(f)$ is the energy density concentrated around f
- E.g., removing frequency component $\equiv$ remove corresponding energy

We omit proof as it is analogous to DFT case. Need to use finite integration region and take limit after exchanging order of integration. Not worth repeating.

Shift $\Leftrightarrow$ modulation

- Two more properties we didn't study for DFTs
  $\Rightarrow$ They (sort of) hold for DFTs, but are difficult to explain

- Time shift $\Rightarrow$ multiplication by complex exponential in frequency
- Multiplication by complex exponential in time $\Rightarrow$ Shift in frequency

- Properties are dual of each other $\Rightarrow$ inverse transform symmetry
  $\Rightarrow$ If one holds the other has to be true

Time shift

- Given signal x and shift τ define shifted signal x_τ $\Rightarrow x_\tau = x(t-\tau)$
- Fourier transform of x is $X = \mathcal{F}(x)$. Transform of x_τ is $X_\tau = \mathcal{F}(x_\tau)$.

Theorem

A time shift of τ units in the time domain is equivalent to multiplication by a complex exponential of frequency $-\tau$ in the frequency domain

$$x_\tau = x(t-\tau) \qquad \Longleftrightarrow \qquad X_\tau(f) = e^{-j2\pi f\tau}X(f)$$

- The phase of $X(f)$ changes, but the modulus remains the same

$$\left|X_\tau(f)\right| = \left|e^{-j2\pi f\tau}X(f)\right| = \left|e^{-j2\pi f\tau}\right| \times \left|X(f)\right| = \left|X(f)\right|$$

- Useful in signal detection $\Rightarrow$ Don't have to compare different shifts

Proof of time shift property

Proof.

- Shifted signal transform $\Rightarrow X_\tau(f) = \int_{-\infty}^{\infty} x(t-\tau)e^{-j2\pi f t}dt$
- Change of variables $u = t - \tau$. Separate exponent in two factors

$$X_\tau(f) = \int_{-\infty}^{\infty} x(u)e^{-j2\pi f(u+\tau)}du = \int_{-\infty}^{\infty} x(u)e^{-j2\pi f\tau}e^{-j2\pi fu}du$$

- Pull the term $e^{-j2\pi f\tau}$ out of the integral. Identify $X(f)$

$$X_\tau(f) = e^{-j2\pi f\tau}\int_{-\infty}^{\infty} x(u)e^{-j2\pi fu}du = e^{-j2\pi f\tau}X(f) \qquad \square$$

Modulation

- For signal x and freq. g define modulated signal $\Rightarrow x_g = e^{-j2\pi gt}x(t)$
- Fourier transform of x is $X = \mathcal{F}(x)$. Transform of x_g is $X_\tau = \mathcal{F}(x_g)$.

Theorem

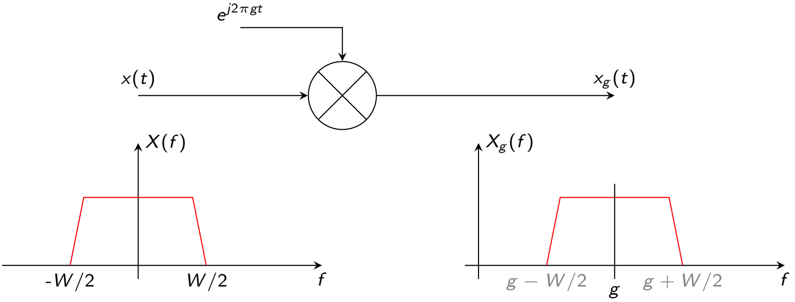
A multiplication by a complex exponential of frequency g in the time domain is equivalent to a shift of g units in the frequency domain

$$x_g = e^{j2\pi gt}x(t) \qquad \Longleftrightarrow \qquad X_g(f) = X(f-g)$$

- Dual of time shift result $\Rightarrow$ Proof not really necessary
- Principle behind transmission of signals on electromagnetic spectrum

Modulation of bandlimited signals

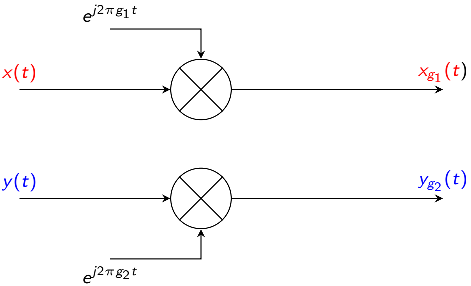
- Signal x has bandwidth W $\Rightarrow X(f) = 0$ for $f \notin [-W/2, W/2]$
- Multiplying by complex exponential shifts spectrum to the right
  $\Rightarrow$ Re-center spectrum at frequency g



- Can recover signal x by multiplying with conjugate frequency $e^{-j2\pi gt}$

Modulation of multiple bandlimited signals

- Modulate two signals with bandwidth W using frequencies g_1 and g_2
  $\Rightarrow$ Spectrum of x recentered at g_1. Spectrum of y recentered at g_2

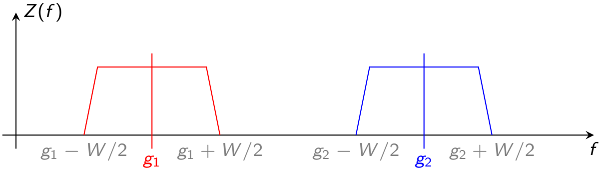


$z(t) = x_{g_1}(t) + y_{g_2}(t)$

- Sum up to construct signal $z(t) = x_{g_1}(t) + y_{g_2}(t)$
  $\Rightarrow$ Can we recover x and y from mixed signal z? $\Rightarrow$ Yes

Spectrum of multiple modulated signals

- No spectral mixing if modulating frequencies satisfy $g_2 - g_1 > W$



- To recover x multiply by conjugate frequency $e^{-j2\pi g_1 t}$
- And eliminated all frequencies outside the interval $[-W/2, W/2]$

- To recover y multiply by conjugate frequency $e^{-j2\pi g_2 t}$
- And eliminated all frequencies outside the interval $[-W/2, W/2]$

Convolution

Continuous time signals

Fourier transform

Inverse Fourier transform

Delta function

Generalized orthogonality

Generalized Fourier transforms

Properties of the Fourier transform

Convolution

Convolution $\Leftrightarrow$ Product

- Both, Fourier transforms and DFTs are:
  - $\Rightarrow$ Conjugate symmetric, linear, & conserve energy
- The Fourier transform also satisfies shift and modulation theorems
  - $\Rightarrow$ They also (sort of) hold for DFTs (although we haven't shown)
  - $\Rightarrow$ As they should, DFTs are close to Fourier transforms
- A sixth property of Fourier transforms, also sort of true for DFTs
  - $\Rightarrow$ Convolution in time equivalent to multiplication in frequency

Convolution

- Given signal x with values $x(t)$ and signal h with values $h(t)$
- Convolution of x with h is the signal $y = x * h$ with values

$$[x * h](t) = y(t) = \int_{-\infty}^{\infty} x(u)h(t-u)\,du$$

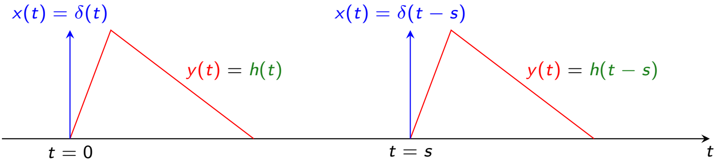
- Operation is commutative $\Rightarrow [x * h] \equiv [h * x]$

$$[h*x](t) = \int_{-\infty}^{\infty} h(u)x(t-u)\,du = \int_{-\infty}^{\infty} h(t-v)x(v)\,dv = [x*h](t)$$

- Still, prefer to interpret roles of x and h as asymmetric $\Rightarrow x$ hits h

Convolution with delta functions

- Convolution with $x(t) = \delta(t) \;\Rightarrow y(t) = \int_{-\infty}^{\infty} \delta(u) h(t-u)\, du = h(t)$
- Hitting h with delta function produces convolution output $y \equiv h$



- Convolution with delayed delta $x(t) = \delta(t-s)$ ($u = s$ in integrand)

$$y(t) = \int_{-\infty}^{\infty} \delta(u-s) h(t-u)\, du = h(t-s)$$

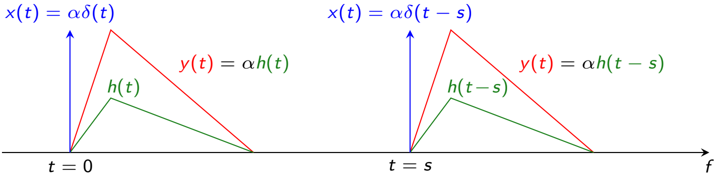
- Hitting h with delayed delta produces delayed h as output

- Convolution with scaled delta function $x(t) = \alpha\delta(t)$

$$y(t) = \int_{-\infty}^{\infty} \alpha\delta(u)h(t-u)\,du = \alpha \int_{-\infty}^{\infty} \delta(u)h(t-u)\,du = \alpha h(t)$$

- Convolution with scaled and delayed delta $x(t) = \alpha\delta(t-s)$

$$y(t) = \int_{-\infty}^{\infty} \alpha\delta(u-s)h(t-u)\,du = \alpha \int_{-\infty}^{\infty} \delta(u-s)h(t-u)\,du = \alpha h(t-s)$$



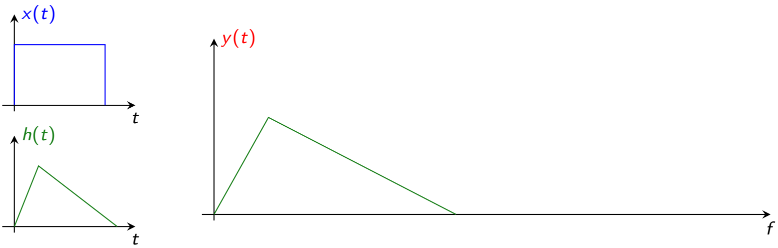
- Convolution with scaled and delayed delta is scaled and delayed h

Interpretation $\Rightarrow$ Scale, Shift, Sum (3S)

- Approximate convolution with Riemann sum (sampling at $u = u_n$)

$$y(t) = \int_{-\infty}^{\infty} x(u)h(t-u)\,du \approx T_s \sum_{n=-\infty}^{\infty} x(u_n)h(t-u_n)$$

- For each u_n $\Rightarrow$ Scale $h(t)$ by $x(u_n)$ to produce $x(u_n)h(t)$
  $\Rightarrow$ Shift to time u_n to produce $x(u_n)h(t-u_n)$
- Sum over all possible u_n $\Rightarrow$ integrate over all u, in the limit



- Linear combination of shifted versions of h with coefficients $x(u)$

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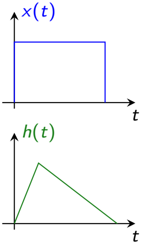

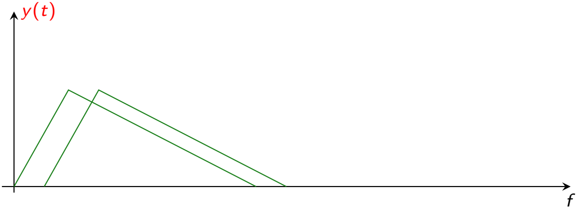

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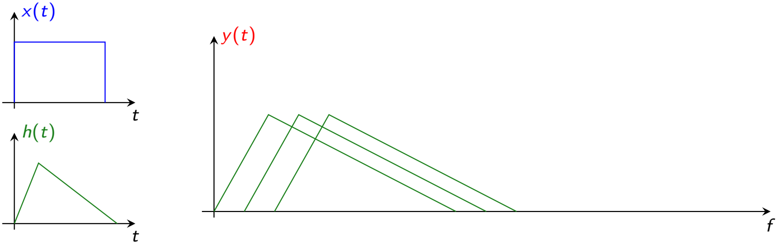

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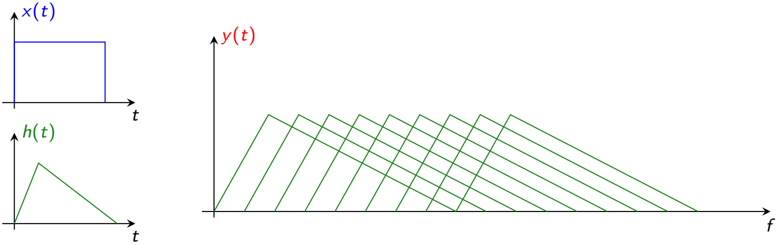
$$y(t) = \int_{-\infty}^{\infty} x(u)h(t-u)\,du \approx T_s \sum_{n=-\infty}^{\infty} x(u_n)h(t-u_n)$$

- For each u_n $\Rightarrow$ Scale $h(t)$ by $x(u_n)$ to produce $x(u_n)h(t)$
  $\Rightarrow$ Shift to time u_n to produce $x(u_n)h(t-u_n)$
- Sum over all possible u_n $\Rightarrow$ integrate over all u, in the limit



- Linear combination of shifted versions of h with coefficients $x(u)$

Time convolution ≡ Frequency multiplication

Theorem (Convolution theorem)

Given signals x and y with transforms $X = \mathcal{F}(x)$ and $Y = \mathcal{F}(y)$. The Fourier transform $Z = \mathcal{F}(z)$ of the convolved signal $z = x * y$ is the product $Z = XY$

$$z = x * y \qquad \Longleftrightarrow \qquad Z = XY$$

- Convolution in time domain ≡ to multiplication in frequency domain
- When we convolve signals x and y in the time domain
  $\Rightarrow$ Their transforms are multiplied in the frequency domain
- When we multiply two transforms in the frequency domain
  $\Rightarrow$ The signals get convolved in the time domain

Proof of convolution theorem

Proof.

- Use the definition of Fourier transform to write the transform of Z as

$$Z(f) = \int_{-\infty}^{\infty} z(t) e^{-j2\pi ft}\, dt$$

- Use the definition of convolution to write the signal z as

$$z(t) = \int_{-\infty}^{\infty} x(u) h(t-u)\, du$$

- Substitute the expression for $z(t)$ into expression for $Z(f)$

$$Y(f) = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x(u) h(t-u)\, du \right) e^{-j2\pi ft}\, dt$$

Proof of convolution theorem

Proof.

- Rewrite the nested integral as a double integral

$$Y(f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(u)h(t-u)e^{-j2\pi ft}\,du\,dt$$

- Make the change of variables $v = t - u$ and write

$$Y(f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(u)h(v)e^{-j2\pi f(u+v)}\,du\,dt$$

- Write $e^{-j2\pi f(u+v)} = e^{-j2\pi fu}e^{-j2\pi fv}$ and reorder terms to obtain

$$Y(f) = \left(\int_{-\infty}^{\infty} x(u)e^{-j2\pi fu}\,du\right)\left(\int_{-\infty}^{\infty} h(v)e^{-j2\pi fv}\,dv\right)$$

- Factors on the right are the Fourier transforms $X(f)$ and $Y(f)$ $\square$

System equivalence

- Convolution in time equivalent to multiplication in frequency
  $\Rightarrow$ Is this useful in any way? $\Rightarrow$ Certainly, few facts are more useful
- Convolution theorem implies that these two systems are equivalent

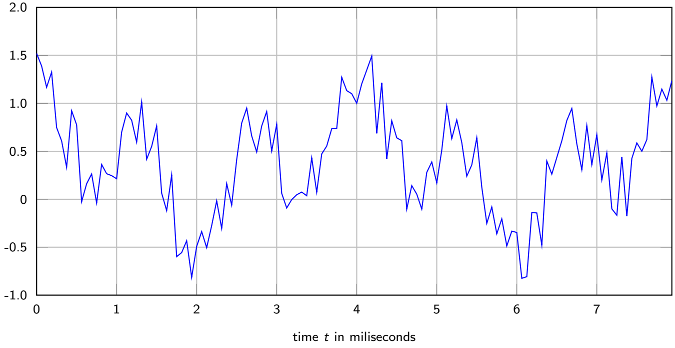
$$\begin{tikzcd}
x \arrow[r] \arrow[d, bend right, "\mathcal{F}"'] & \boxed{h} \arrow[r] \arrow[d, bend right, "\mathcal{F}"'] & y = x * h \arrow[d, bend right, "\mathcal{F}"'] \\
X \arrow[r] \arrow[u, bend right, "\mathcal{F}^{-1}"'] & \boxed{H} \arrow[r] \arrow[u, bend right, "\mathcal{F}^{-1}"'] & Y = HX \arrow[u, bend right, "\mathcal{F}^{-1}"']
\end{tikzcd}$$

- The lower path for design, the upper path for implementation

The signal and the noise

- There is signal and noise, but what is signal and what is noise?
- We already know answer $\Rightarrow$ Signal discernible in frequency domain

Original signal $x(t)$. It moves randomly, but not that much



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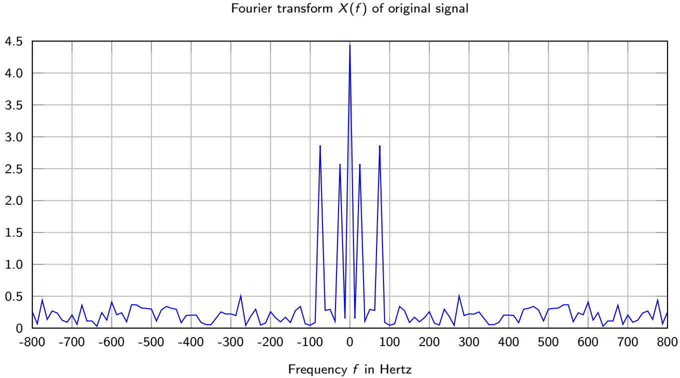
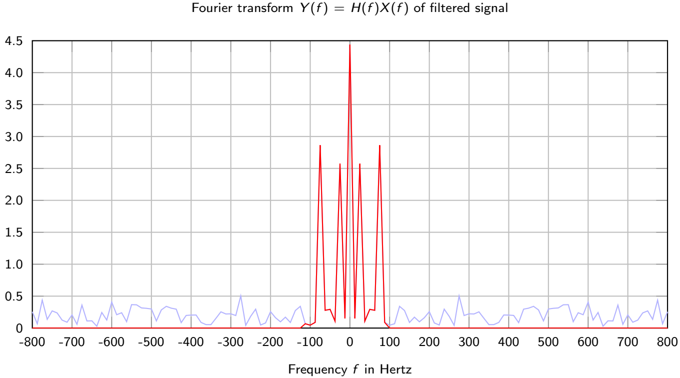


- Filter out all frequencies above 100Hz (and below -100Hz)

Noise removal – Low pass filter design

- Multiply spectrum with low pass filter $H(f) = \sqcap_W(f)$ with $W = 200$Hz
  $\Rightarrow$ Only frequencies between $\pm W/2 = \pm 100$Hz are retained



- This spectral operation does separate signal from noise

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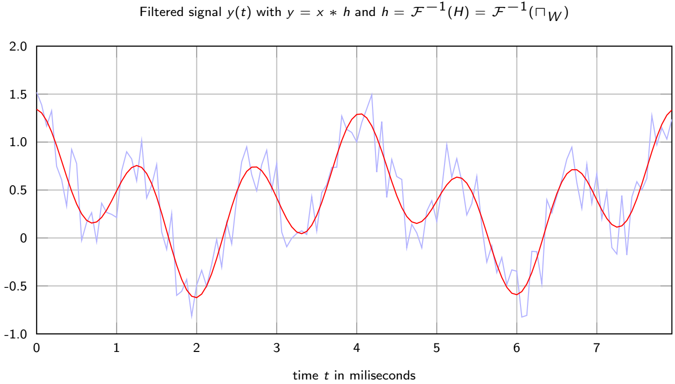

- This spectral operation does separate signal from noise

Noise removal – Low pass filter implementation

- We can implement filtering in the frequency domain
  - $\Rightarrow$ Sample $\Rightarrow$ DFT $\Rightarrow$ Multiply by $H(f) = \sqcap_W(f)$ $\Rightarrow$ iDFT

$$
\begin{tikzcd}
X \arrow[r] & \boxed{H(f) = \sqcap_W(f)} \arrow[r] & Y = HX \arrow[d, bend left, "\mathcal{F}^{-1}"] \\
x \arrow[u, bend left, "\mathcal{F}"] \arrow[r] & \boxed{h(t) = W\text{sinc}(\pi W t)} \arrow[r] & y = x * h
\end{tikzcd}
$$

- We can also implement filtering in the time domain
  - $\Rightarrow$ Inverse transform of $\sqcap_W(f)$ is $h(t) = W\text{sinc}(\pi W t)$
  - $\Rightarrow$ Sample (or not) $\Rightarrow$ Implement convolution with $h(t)$